1

Basic notions

In this introductory chapter some mathematical notions are presented rapidly, which lie at the heart of the study of Mathematical Analysis. Most should already be known to the reader, perhaps in a more thorough form than in the following presentation. Other concepts may be completely new, instead. The treatise aims at fixing much of the notation and mathematical symbols frequently used in the sequel.

1.1 Sets

We shall denote **sets** mainly by upper case letters $X, Y, \ldots$, while for the members or elements of a set lower case letters $x, y, \ldots$ will be used. When an element x is in the set X one writes $x \in X$ ('x is an element of X', or 'the element x belongs to the set X'), otherwise the symbol $x \notin X$ is used.

The majority of sets we shall consider are built starting from sets of numbers. Due to their importance, the main sets of numbers deserve special symbols, namely:

$\mathbb{N}$ = set of natural numbers
$\mathbb{Z}$ = set of integer numbers
$\mathbb{Q}$ = set of rational numbers
$\mathbb{R}$ = set of real numbers
$\mathbb{C}$ = set of complex numbers.

The definition and main properties of these sets, apart from the last one, will be briefly recalled in Sect. 1.3. Complex numbers will be dealt with separately in Sect. 8.3.

Let us fix a non-empty set X, considered as *ambient set*. A **subset** A of X is a set all of whose elements belong to X; one writes $A \subseteq X$ ('A is contained, or included, in X') if the subset A is allowed to possibly coincide with X, and $A \subset X$ ('A is properly contained in X') in case A is a *proper* subset of X, that

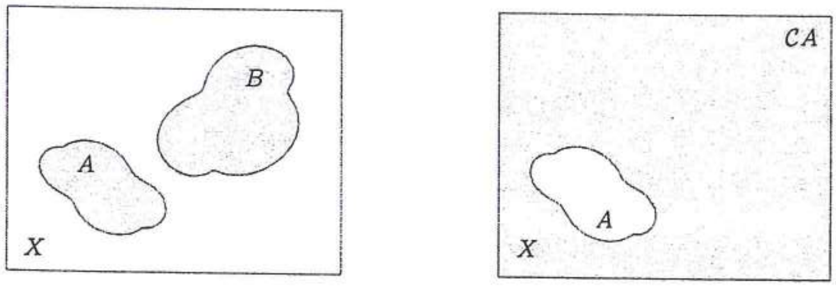

Figure 1.1. Venn diagrams (left) and complement (right)

is, if it does not exhaust the whole X. From the intuitive point of view it may be useful to represent subsets as bounded regions in the plane using the so-called *Venn diagrams* (see Fig. 1.1, left).

A subset can be described by listing the elements of X which belong to it

$$A = \{x, y, \dots, z\};$$

the order in which elements appear is not essential. This clearly restricts the use of such notation to subsets with few elements. More often the notation

$$A = \{x \in X \mid p(x)\} \qquad \text{or} \qquad A = \{x \in X \,:\, p(x)\}$$

will be used (read 'A is the subset of elements x of X such that the condition $p(x)$ holds'); $p(x)$ denotes the *characteristic property* of the elements of the subset, i.e., the condition that is valid for the elements of the subset only, and not for other elements. For example, the subset A of natural numbers smaller or equal than 4 may be denoted

$$A = \{0, 1, 2, 3, 4\} \qquad \text{or} \qquad A = \{x \in \mathbb{N} \mid x \leq 4\}.$$

The expression $p(x) =$'$x \leq 4$' is an example of *predicate*, which we will return to in the following section.

The collection of all subsets of a given set X forms the **power set** of X, and is denoted by $\mathcal{P}(X)$. Obviously $X \in \mathcal{P}(X)$. Among the subsets of X there is the **empty set**, the set containing no elements. It is usually denoted by the symbol $\emptyset$, so $\emptyset \in \mathcal{P}(X)$. All other subsets of X are proper and non-empty.

Consider for instance $X = \{1, 2, 3\}$ as ambient set. Then

$$\mathcal{P}(X) = \{\,\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}.$$

Note that X contains 3 elements (it has *cardinality* 3), while $\mathcal{P}(X)$ has $8 = 2^3$ elements, hence has cardinality 8. In general if a finite set (a set with a finite number of elements) has cardinality n, the power set of X has cardinality 2^n.

Starting from one or more subsets of X, one can define new subsets by means of set-theoretical operations. The simplest operation consists in taking the complement: if A is a subset of X, one defines the **complement** of A (in X) to be the subset

$$\boxed{\mathcal{C}A = \{x \in X \mid x \notin A\}}$$

made of all elements of X not belonging to A (Fig. 1.1, right).

Sometimes, in order to underline that complements are taken with respect to the ambient space X, one uses the more precise notation $\mathcal{C}_X A$. The following properties are immediate:

$$\mathcal{C}X = \emptyset, \qquad \mathcal{C}\emptyset = X, \qquad \mathcal{C}(\mathcal{C}A) = A.$$

For example, if $X = \mathbb{N}$ and A is the subset of *even* numbers (multiples of 2), then $\mathcal{C}A$ is the subset of *odd* numbers.

Given two subsets A and B of X, one defines **intersection** of A and B the subset

$$\boxed{A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}}$$

containing the elements of X that belong to both A and B, and **union** of A and B the subset

$$\boxed{A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}}$$

made of the elements that are either in A or in B (this is meant non-exclusively, so it includes elements of $A \cap B$), see Fig. 1.2.

We recall some properties of these operations.

i) Boolean properties:

$$A \cap \mathcal{C}A = \emptyset, \qquad A \cup \mathcal{C}A = X;$$

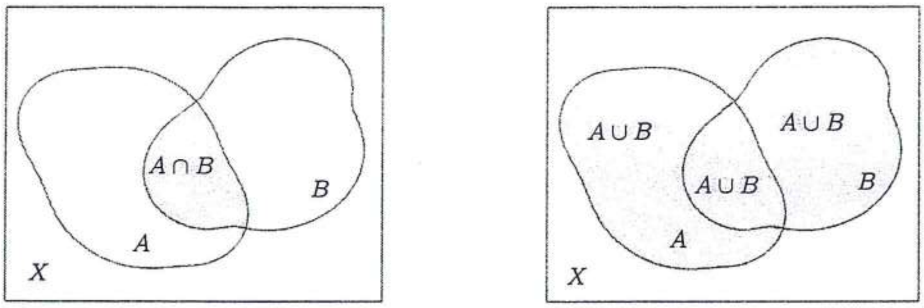

Figure 1.2. Intersection and union of sets

ii) commutative, associative and distributive properties:

$$A \cap B = B \cap A, \qquad A \cup B = B \cup A,$$
$$(A \cap B) \cap C = A \cap (B \cap C), \qquad (A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \qquad (A \cup B) \cap C = (A \cap C) \cup (B \cap C);$$

iii) De Morgan laws:

$$\mathcal{C}(A \cap B) = \mathcal{C}A \cup \mathcal{C}B, \qquad \mathcal{C}(A \cup B) = \mathcal{C}A \cap \mathcal{C}B.$$

Notice that the condition $A \subseteq B$ is equivalent to $A \cap B = A$, or $A \cup B = B$.

There are another couple of useful operations. The first is the **difference** between a subset A and a subset B, sometimes called **relative complement of B in A**

$$\boxed{A \setminus B = \{x \in A \mid x \notin B\} = A \cap \mathcal{C}B}$$

(read 'A minus B'), which selects the elements of A that do not belong to B. The second operation is the **symmetric difference** of the subsets A and B

$$\boxed{A \,\Delta\, B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B),}$$

which picks out the elements belonging either to A or B, but not both (Fig. 1.3).

For example, let $X = \mathbb{N}$, A be the set of even numbers and $B = \{n \in \mathbb{N} \mid n \leq 10\}$ the set of natural numbers smaller or equal than 10. Then $B \setminus A = \{1, 3, 5, 7, 9\}$ is the set of odd numbers smaller than 10, $A \setminus B$ is the set of even numbers larger than 10, and $A \Delta B$ is the union of the latter two.

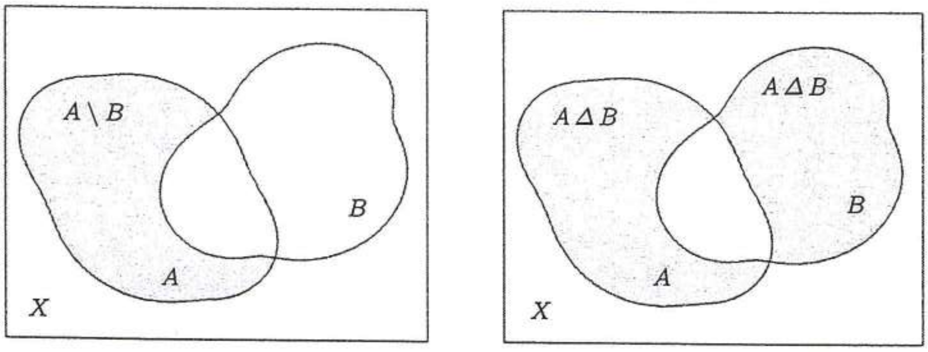

Figure 1.3. The difference $A \setminus B$ (left) and the symmetric difference $A \,\Delta\, B$ (right) of two sets

$p(x)$ = 'x is strictly less than 7' for example, yields the false formula '$\forall x \in \mathbb{N},\ p(x)$' (since $p(8)$ is false, for example), while '$\exists x \in \mathbb{N},\ p(x)$' is true (e.g., $x = 6$ satisfies the assertion).

The effect of negation on a quantified predicate must be handled with attention. Suppose for instance x indicates the generic student of the Polytechnic, and let $p(x)$ = 'x is an Italian citizen'. The formula '$\forall x,\ p(x)$' ('every student of the Polytechnic has Italian citizenship') is false. Therefore its negation '$\neg(\forall x,\ p(x))$' is true, but beware: the latter does not state that all students are foreign, rather that 'there is at least one student who is not Italian'. Thus the negation of '$\forall x,\ p(x)$' is '$\exists x,\ \neg p(x)$'. We can symbolically write

$$\boxed{\neg(\forall x,\ p(x)) \iff \exists x,\ \neg p(x).}$$

Similarly, it is not hard to convince oneself of the logic equivalence

$$\boxed{\neg(\exists x,\ p(x)) \iff \forall x,\ \neg p(x).}$$

If a predicate depends upon two or more arguments, each of them may be quantified. Yet the *order* in which the quantifiers are written can be essential. Namely, two quantifiers of the same type (either universal or existential) can be swapped without modifying the truth value of the formula; in other terms

$$\boxed{\begin{aligned} \forall x\, \forall y,\ p(x,y) &\iff \forall y\, \forall x,\ p(x,y), \\ \exists x\, \exists y,\ p(x,y) &\iff \exists y\, \exists x,\ p(x,y). \end{aligned}}$$

On the contrary, exchanging the places of different quantifiers usually leads to different formulas, so one should be very careful when ordering quantifiers.

As an example, consider the predicate $p(x,y)$ = '$x \geq y$', with x, y varying in the set of natural numbers. The formula '$\forall x\, \forall y,\ p(x,y)$' means 'given any two natural numbers, each one is greater or equal than the other', clearly a false statement. The formula '$\forall x\, \exists y,\ p(x,y)$', meaning 'given any natural number x, there is a natural number y smaller or equal than x', is true, just take $y = x$ for instance. The formula '$\exists x\, \forall y,\ p(x,y)$' means 'there is a natural number x greater or equal than each natural number', and is false: each natural number x admits a successor $x + 1$ which is strictly bigger than x. Eventually, '$\exists x\, \exists y,\ p(x,y)$' ('there are at least two natural numbers such that one is bigger or equal than the other') holds trivially.

1.3 Sets of numbers

Let us briefly examine the main sets of numbers used in the book. The discussion is on purpose not exhaustive, since the main properties of these sets should already be known to the reader.

The set $\mathbb{N}$ of natural numbers. This set has the numbers $0, 1, 2, \ldots$ as elements. The operations of sum and product are defined on $\mathbb{N}$ and enjoy the well-known commutative, associative and distributive properties. We shall indicate by $\mathbb{N}_+$ the set of natural numbers different from 0

$$\mathbb{N}_+ = \mathbb{N} \setminus \{0\}.$$

A natural number n is usually represented in *base 10* by the expansion $n = c_k 10^k + c_{k-1} 10^{k-1} + \cdots + c_1 10 + c_0$, where the c_i's are natural numbers from 0 to 9 called *decimal digits*; the expression is unique if one assumes $c_k \neq 0$ when $n \neq 0$. We shall write $n = (c_k c_{k-1} \ldots c_1 c_0)_{10}$, or more easily $n = c_k c_{k-1} \ldots c_1 c_0$. Any natural number ≥ 2 may be taken as base, instead of 10; a rather common alternative is 2, known as *binary base.*

Natural numbers can also be represented geometrically as points on a straight line. For this it is sufficient to fix a first point O on the line, called *origin*, and associate it to the number 0, and then choose another point P different from O, associated to the number 1. The direction of the line going from O to P is called *positive direction*, while the length of the segment OP is taken as *unit* for measurements. By marking multiples of OP on the line in the positive direction we obtain the points associated to the natural numbers (see Fig. 1.4).

The set $\mathbb{Z}$ of integer numbers. This set contains the numbers $0, +1, -1, +2, -2, \ldots$ (called integers). The set $\mathbb{N}$ can be identified with the subset of $\mathbb{Z}$ consisting of $0, +1, +2, \ldots$ The numbers $+1, +2, \ldots$ $(-1, -2, \ldots)$ are said *positive integers* (resp. *negative integers*). Sum and product are defined in $\mathbb{Z}$, together with the difference, which is the inverse operation to the sum.

An integer can be represented in decimal base $z = \pm c_k c_{k-1} \ldots c_1 c_0$. The geometric picture of negative integers extends that of the natural numbers to the left of the origin (Fig. 1.4).

The set $\mathbb{Q}$ of rational numbers. A rational number is the quotient, or ratio, of two integers, the second of which (denominator) is non-zero. Without loss of generality one can assume that the denominator is positive, whence each rational number, or rational for simplicity, is given by

$$r = \frac{z}{n}, \qquad \text{with } z \in \mathbb{Z} \text{ and } n \in \mathbb{N}_+.$$

Moreover, one may also suppose the fraction is reduced, that is, z and n have no common divisors. In this way the set $\mathbb{Z}$ is identified with the subset of rationals

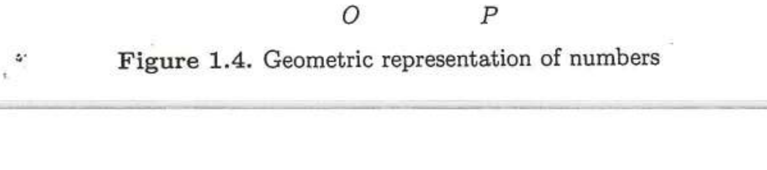

Figure 1.4. Geometric representation of numbers

whose denominator is 1. Besides sum, product and difference, the operation of division between two rationals is defined on $\mathbb{Q}$, so long as the second rational is other than 0. This is the inverse to the product.

A rational number admits a representation in base 10 of the kind $r = \pm c_k c_{k-1} \cdots c_1 c_0 . d_1 d_2 \cdots$, corresponding to

$$r = \pm(c_k 10^k + c_{k-1} 10^{k-1} + \cdots + c_1 10 + c_0 + d_1 10^{-1} + d_2 10^{-2} + \cdots).$$

The sequence of digits $d_1, d_2, \ldots$ written after the dot satisfies one and only one of the following properties: i) all digits are 0 from a certain subscript $i \geq 1$ onwards (in which case one has a *finite decimal expansion*; usually the zeroes are not written), or ii) starting from a certain point, a finite sequence of numbers not all zero – called *period* – repeats itself over and over (*infinite periodic decimal expansion*; the period is written once with a line drawn on top). For example the following expressions are decimal expansions of rational numbers

$$-\frac{35163}{100} = -351.6300 \cdots = -371.63 \quad \text{and} \quad \frac{11579}{925} = 12.51783783 \cdots = 12.51\overline{783}.$$

The expansion of certain rationals is not unique. If a rational number has a finite expansion in fact, then it also has a never-ending periodic one obtained from the former by reducing the right-most non-zero decimal digit by one unit, and adding the period 9. The expansions 1.0 and $0.\overline{9}$ define the same rational number 1; similarly, 8.357 and $8.356\overline{9}$ are equivalent representations of $\frac{4120}{493}$.

The geometric representation of a rational $r = \pm\frac{m}{n}$ is obtained by subdividing the segment OP in n equal parts and copying the subsegment m times in the positive or negative direction, according to the sign of r (see again Fig. 1.4).

The set $\mathbb{R}$ of real numbers. Not every point on the line corresponds to a rational number in the above picture. This means that not all segments can be measured by multiples and sub-multiples of the unit of length, irrespective of the choice of this unit.

It has been known since the ancient times that the diagonal of a square is not *commensurable* with the side, meaning that the length d of the diagonal is not a rational multiple of the side's length ℓ. To convince ourselves about this fact recall Pythagoras's Theorem. It considers any of the two triangles in which the diagonal splits the square (Fig. 1.5), and states that

$$d^2 = \ell^2 + \ell^2, \qquad \text{i.e.,} \qquad d^2 = 2\ell^2.$$

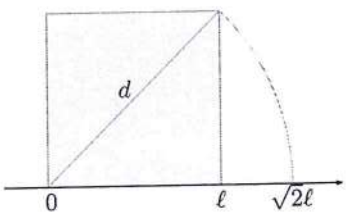

Figure 1.5. Square with side ℓ and its diagonal

Calling p the ratio between the lengths of diagonal and side, we square $d = p\ell$ and substitute in the last relation to obtain $p^2 = 2$. The number p is called the *square root* of 2 and it is indicated by the symbol $\sqrt{2}$.

Property 1.1 *If the number p satisfies $p^2 = 2$, it must be non-rational.*

Proof. By contradiction: suppose there exist two integers m and n, necessarily non-zero, such that $p = \frac{m}{n}$. Assume m, n are relatively prime. Taking squares we obtain $\frac{m^2}{n^2} = 2$, hence $m^2 = 2n^2$. Thus m^2 is even, which is to say that m is even. For a suitable natural number k then, $m = 2k$. Using this in the previous relation yields $4k^2 = 2n^2$, i.e., $n^2 = 2k^2$. Then n^2, whence also n, is even. But this contradicts the fact that m and n have no common factor, which comes from the assumption that p is rational. □

Another relevant example of incommensurable lengths, known for centuries, pertains to the length of a circle measured with respect to the diameter. In this case as well, one can prove that the lengths of circumference and diameter are not commensurable because the proportionality factor, known by the symbol π, cannot be a rational number.

The set of real numbers is an extension of the rationals and provides a *mathematical model of the straight line*, in the sense that each real number x can be associated to a point P on the line uniquely, and vice versa. The former is called the *coordinate* of P. There are several equivalent ways of constructing such extension. Without going into details, we merely recall that real numbers give rise to any possible decimal expansion. Real numbers that are not rational, called *irrational*, are characterised by having a *non-periodic infinite* decimal expansion, like

$$\sqrt{2} = 1.4142135623731\cdots \qquad \text{and} \qquad \pi = 3.14159265358979\cdots$$

Rather than the actual construction of the set $\mathbb{R}$, what is more interesting to us are the properties of real numbers, which allow one to work with the reals. Among these properties, we recall some of the most important ones.

i) *The arithmetic operations defined on the rationals extend to the reals* with similar properties.
ii) *The order relation $x < y$ of the rationals extends to the reals*, again with similar features. We shall discuss this matter more deeply in the following Sect. 1.3.1.
iii) *Rational numbers are dense in the set of real numbers.* This means there are infinitely many rationals sitting between any two real numbers. It also implies that each real number can be approximated by a rational number as well as we please. If for example $r = c_k c_{k-1} \cdots c_1 c_0.d_1 d_2 \cdots d_i d_{i+1} \cdots$ has a non-periodic infinite decimal expansion, we can approximate it by the rational $q_i = c_k c_{k-1} \cdots c_1 c_0.d_1 d_2 \cdots d_i$ obtained by ignoring all decimal digits past the ith one; as i increases, the approximation of r will get better and better.

iv) *The set of real numbers is complete.* Geometrically speaking, this is equivalent to asking that each point on the line is associated to a unique real number, as already mentioned. Completeness guarantees for instance the existence of the square root of 2, i.e., the solvability in $\mathbb{R}$ of the equation $x^2 = 2$, as well as of infinitely many other equations, algebraic or not. We shall return to this point in Sect. 1.3.2.

1.3.1 The ordering of real numbers

Non-zero real numbers are either positive or negative. Positive reals form the subset $\mathbb{R}_+$, negative reals the subset $\mathbb{R}_-$. We are thus in presence of a partition $\mathbb{R} = \mathbb{R}_- \cup \{0\} \cup \mathbb{R}_+$. The set

$$\mathbb{R}_* = \{0\} \cup \mathbb{R}_+$$

of non-negative reals will also be needed. Positive numbers correspond to points on the line lying at the right – with respect to the positive direction – of the origin.

Instead of $x \in \mathbb{R}_+$, one simply writes $x > 0$ ('x is bigger, or larger, than 0'); similarly, $x \in \mathbb{R}_*$ will be expressed by $x \geq 0$ ('x is bigger or equal than 0'). Therefore an order relation is defined by

$$x < y \qquad \Longleftrightarrow \qquad y - x > 0.$$

This is a *total* ordering, i.e., given any two distinct reals x and y, one (and only one) of the following holds: either $x < y$ or $y < x$. From the geometrical point of view the relation $x < y$ tells that the point with coordinate x is placed at the left of the point with coordinate y. Let us also define

$$\boxed{x \leq y \qquad \Longleftrightarrow \qquad x < y \quad \text{or} \quad x = y.}$$

Clearly, $x < y$ implies $x \leq y$. For example the relations $3 \leq 7$ and $7 \leq 7$ are true, whereas $3 \leq 2$ is not.

The order relation $\leq$ (or $<$) interacts with the algebraic operations of sum and product as follows:

$$\boxed{\text{if } x \leq y \text{ and } z \text{ is any real number, then } x + z \leq y + z}$$

(adding the same real number to both sides of an inequality leaves the latter unchanged);

$$\boxed{\text{if } x \leq y \ \text{ and } \ \text{if} \begin{cases} z \geq 0, & \text{then } \ xz \leq yz, \\ z < 0, & \text{then } \ xz \geq yz \end{cases}}$$

(multiplying by a non-negative number both sides of an inequality does not alter it, while if the number is negative it inverts the inequality). Example: multiplying by -1 the inequality $-3 \leq 2$ gives $-2 \leq 3$. The latter property implies the well-known

sign rule: the product of two numbers with alike signs is positive, the product of two numbers of different sign is negative.

Absolute value. Let us introduce now a simple yet important notion. Given a real number x, one calls *absolute value* of x the real number

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Thus $|x| \geq 0$ for any x in $\mathbb{R}$. For instance $|5| = 5$, $|0| = 0$, $|-5| = 5$. Geometrically, $|x|$ represents the distance from the origin of the point with coordinate x; thus, $|x - y| = |y - x|$ is the distance between the two points of coordinates x and y.

The following relations, easy to prove, will be useful

$$|x + y| \leq |x| + |y|, \qquad \text{for all } x, y \in \mathbb{R} \tag{1.1}$$

(called *triangle inequality*) and

$$|xy| = |x||y|, \qquad \text{for all } x, y \in \mathbb{R}.$$

Throughout the text we shall solve equations and inequalities involving absolute values. Let us see the simplest ones. According to the definition,

$$|x| = 0$$

has the unique solution $x = 0$. If a is any number > 0, the equation

$$|x| = a$$

has two solutions $x = a$ and $x = -a$, so

$$|x| = a \qquad \Longleftrightarrow \qquad x = \pm a, \qquad \forall a \geq 0.$$

In order to solve

$$|x| \leq a, \qquad \text{where } a \geq 0,$$

consider first the solutions $x \geq 0$, for which $|x| = x$, so that now the inequality reads $x \leq a$; then consider $x < 0$, in which case $|x| = -x$, and solve $-x \leq a$, or $-a \leq x$. To summarise, the solutions are real numbers x satisfying $0 \leq x \leq a$ or $-a \leq x < 0$, which may be written in a shorter way as

$$|x| \leq a \qquad \Longleftrightarrow \qquad -a \leq x \leq a. \tag{1.2}$$

Similarly, it is easy to see that if $b \geq 0$,

$$\boxed{|x| \geq b \quad \iff \quad x \leq -b \ \text{ or } \ x \geq b.} \tag{1.3}$$

The slightly more general inequality

$$|x - x_0| \leq a,$$

where $x_0 \in \mathbb{R}$ is fixed and $a \geq 0$, is equivalent to $-a \leq x - x_0 \leq a$; adding x_0 gives

$$\boxed{|x - x_0| \leq a \quad \iff \quad x_0 - a \leq x \leq x_0 + a.} \tag{1.4}$$

In all examples we can replace the symbol $\leq$ by $<$ and the conclusions hold.

Intervals. The previous discussion shows that Mathematical Analysis often deals with subsets of $\mathbb{R}$ whose elements lie between two fixed numbers. They are called intervals.

Definition 1.2 *Let a and b be real numbers such that $a \leq b$. The* **closed interval** *with end-points a, b is the set*

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

If $a < b$, one defines **open interval** *with end-points a, b the set*

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

An equivalent notation is $]a, b[$.

If one includes only one end-point, then the interval with end-points a, b

$$\boxed{[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}}$$

is called *half-open on the right*, while

$$\boxed{(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}.}$$

is *half-open on the left*.

Figure 1.6. Geometric representation of the closed interval $[a, b]$ (left) and of the open interval (a, b) (right)

Example 1.3

Describe the set A of elements $x \in \mathbb{R}$ such that

$$2 \leq |x| < 5.$$

Because of (1.2) and (1.3), we easily have

$$A = (-5, -2] \cup [2, 5).$$ □

Intervals defined by a single inequality are useful, too. Define

$$[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}, \qquad (a, +\infty) = \{x \in \mathbb{R} \mid a < x\},$$

and

$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}, \qquad (-\infty, b) = \{x \in \mathbb{R} \mid x < b\}.$$

The symbols $-\infty$ and $+\infty$ do not indicate real numbers; they allow to extend the ordering of the reals with the convention that $-\infty < x$ and $x < +\infty$ for all $x \in \mathbb{R}$. Otherwise said, the condition $a \leq x$ is the same as $a \leq x < +\infty$, so the notation $[a, +\infty)$ is consistent with the one used for real end-points. Sometimes it is convenient to set

$$(-\infty, +\infty) = \mathbb{R}.$$

In general one says that an interval I is **closed** if it contains its end-points, **open** if the end-points are not included. All points of an interval, apart from the end-points, are called **interior points**.

Bounded sets. Let us now discuss the notion of boundedness of a set.

Definition 1.4 *A subset A of $\mathbb{R}$ is called* **bounded from above** *if there exists a real number b such that*

$$x \leq b, \qquad \textit{for all } x \in A.$$

Any b with this property is called **an upper bound** *of A.*
The set A is **bounded from below** *if there is a real number a with*

$$a \leq x, \qquad \textit{for all } x \in A.$$

Every a satisfying this relation is said **a lower bound** *of A.*
At last, one calls A **bounded** *if it is bounded from above and below.*

In terms of intervals, a set is bounded from above if it is contained in an interval of the sort $(-\infty, b]$ with $b \in \mathbb{R}$, and bounded if it is contained in an interval $[a, b]$ for some $a, b \in \mathbb{R}$. It is not difficult to show that A is bounded if and only if there exists a real $c > 0$ such that

$$|x| \leq c, \qquad \text{for all } x \in A.$$

Examples 1.5

i) The set $\mathbb{N}$ is bounded from below (each number $a \leq 0$ is a lower bound), but not from above: in fact, the so-called **Archimedean property** holds: *for any real* $b > 0$, *there exists a natural number* n *with*

$$n > b. \tag{1.5}$$

ii) The interval $(-\infty, 1]$ is bounded from above, not from below. The interval $(-5, 12)$ is bounded.

iii) The set

$$A = \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\} \tag{1.6}$$

is bounded, in fact $0 \leq \dfrac{n}{n+1} < 1$ for any $n \in \mathbb{N}$.

iv) The set $B = \{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded. Taking x such that $|x| > \frac{3}{2}$ for example, then $x^2 > \frac{9}{4} > 2$, so $x \notin B$. Thus $B \subset [-\frac{3}{2}, \frac{3}{2}]$. □

Definition 1.6 *A set* $A \subset \mathbb{R}$ **admits a maximum** *if an element* $x_M \in A$ *exists such that*

$$x \leq x_M, \qquad \textit{for any } x \in A.$$

The element x_M *(necessarily unique) is the* **maximum of the set** A *and one denotes it by* $x_M = \max A$.
The **minimum** *of a set* A, *denoted by* $x_m = \min A$, *is defined in a similar way.*

A set admitting a maximum must be bounded from above: the maximum is an upper bound for the set, actually *the smallest of all possible upper bounds*, as we shall prove. The opposite is not true: a set can be bounded from above but not admit a maximum, like the set A of (1.6). We know already that 1 is an upper bound for A. Among all upper bounds, 1 is privileged, being *the smallest upper bound.* To convince ourselves of this fact, let us show that each real number $r < 1$ is not an upper bound, i.e., there is a natural number n such that

$$\frac{n}{n+1} > r.$$

The inequality is equivalent to $\dfrac{n+1}{n} < \dfrac{1}{r}$, hence $1 + \dfrac{1}{n} < \dfrac{1}{r}$, or $\dfrac{1}{n} < \dfrac{1-r}{r}$. This is to say $n > \dfrac{r}{1-r}$, and the existence of such n follows from property (1.5). So, 1 is the smallest upper bound of A, yet not the maximum, for $1 \notin A$: there is no natural number n such that $\dfrac{n}{n+1} = 1$. One calls 1 the *supremum*, or *least upper bound*, of A and writes $1 = \sup A$.

Analogously, 2 is the smallest of upper bounds of the interval $I = (0, 2)$, but it does not belong to I. Thus 2 is the supremum, or least upper bound, of I, $2 = \sup I$.

Definition 1.7 *Let $A \subset \mathbb{R}$ be bounded from above. The* **supremum** *or* **least upper bound of** *A is the smallest of all upper bounds of A, denoted by* $\sup A$. *If $A \subset \mathbb{R}$ is bounded from below, one calls* **infimum** *or* **greatest lower bound of** *A the largest of all lower bounds of A. This is denoted by* $\inf A$.

The number $s = \sup A$ is characterised by two conditions:

$$\begin{array}{ll} i) & x \leq s \text{ for all } x \in A; \\ ii) & \text{for any real } r < s, \text{ there is an } x \in A \text{ with } x > r. \end{array} \tag{1.7}$$

While *i)* tells that s is an upper bound for A, according to *ii)* each number smaller than s *is not* an upper bound for A, rendering s the smallest among all upper bounds.

The two conditions (1.7) must be fulfilled in order to show that a given number is the supremum of a set. That is precisely what we did to claim that 1 was the supremum of (1.6).

The notion of supremum generalises that of maximum of a set. It is immediate to see that if a set admits a maximum, this maximum must be the supremum as well.

If a set A is not bounded from above, one says that its supremum is $+\infty$, i.e., one defines

$$\sup A = +\infty.$$

Similarly, $\inf A = -\infty$ for a set A not bounded from below.

1.3.2 Completeness of $\mathbb{R}$

The property of completeness of $\mathbb{R}$ may be formalised in several equivalent ways. The reader should have already come across (*Dedekind's*) *separability axiom*: decomposing $\mathbb{R}$ into the union of two disjoint subsets C_1 and C_2 (the pair (C_1, C_2) is called a *cut*) so that each element of C_1 is smaller or equal than every element in C_2, there exists a (unique) separating element $s \in \mathbb{R}$:

$$x_1 \leq s \leq x_2, \qquad \forall x_1 \in C_1,\ \forall x_2 \in C_2.$$

An alternative formulation of completeness involves the notion of supremum of a set: *every bounded set from above admits a supremum in* $\mathbb{R}$, i.e., there is a real number smaller or equal than all upper bounds of the set.

With the help of this property one can prove, for example, the existence in $\mathbb{R}$ of the square root of 2, hence of a number p (> 0) such that $p^2 = 2$. Going back to Example 1.5 iv), the completeness of the reals ensures that the bounded set $B = \{x \in \mathbb{Q} \mid x^2 < 2\}$ has a supremum, say p. Using the properties of $\mathbb{R}$ it is possible to show that $p^2 < 2$ cannot occur, otherwise p would not be an upper bound for B, and neither $p^2 > 2$ holds, for p would not be the least of all upper bounds. Thus necessarily $p^2 = 2$. Note that B, albeit contained in $\mathbb{Q}$, is not allowed to have a rational upper bound, because $p^2 = 2$ prevents p from being rational (Property 1.1).

This example explains why the completeness of $\mathbb{R}$ lies at the core of the possibility to solve in $\mathbb{R}$ many remarkable equations. We are thinking in particular about the family of algebraic equations

$$x^n = a, \tag{1.8}$$

where $n \in \mathbb{N}_+$ and $a \in \mathbb{R}$, for which it is worth recalling the following known fact.

Property 1.8 *i) Let $n \in \mathbb{N}_+$ be odd. Then for any $a \in \mathbb{R}$ equation (1.8) has exactly one solution in $\mathbb{R}$, denoted by $x = \sqrt[n]{a}$ or $x = a^{1/n}$ and called the* **nth root of** *a.*

ii) Let $n \in \mathbb{N}_+$ be even. For any $a > 0$ equation (1.8) has two real solutions with the same absolute value but opposite signs; when $a = 0$ there is one solution $x = 0$ only; for $a < 0$ there are no solutions in $\mathbb{R}$. The non-negative solution is indicated by $x = \sqrt[n]{a}$ or $x = a^{1/n}$, and called the **nth (arithmetic) root of** *a.*

1.4 Factorials and binomial coefficients

We introduce now some noteworthy integers that play a role in many areas of Mathematics.

Given a natural number $n \geq 1$, the product of all natural numbers between 1 and n goes under the name of **factorial of** n and is indicated by $n!$ (read 'n factorial'). Out of conveniency one sets $0! = 1$. Thus

$$0! = 1, \qquad 1! = 1, \qquad n! = 1 \cdot 2 \cdot \ldots \cdot n = (n-1)!\, n \quad \text{for } n \geq 2. \tag{1.9}$$

Factorials grow extremely rapidly as n increases; for instance $5! = 120$, $10! = 3628800$ and $100! > 10^{157}$.

Example 1.9

Suppose we have $n \geq 2$ balls of different colours in a box. In how many ways can we extract the balls from the box?

When taking the first ball we are making a choice among the n balls in the box; the second ball will be chosen among the $n-1$ balls left, the third one among $n-2$ and so on. Altogether we have $n(n-1)\cdot\ldots\cdot 2\cdot 1 = n!$ different ways to extract the balls: $n!$ represents the number of arrangements of n distinct objects in a sequence, called **permutations** of n ordered objects.
If we stop after k extractions, $0 < k < n$, we end up with $n(n-1)\ldots(n-k+1)$ possible outcomes. The latter expression, also written as $\frac{n!}{(n-k)!}$, is the number of possible **permutations of n distinct objects in sequences of k objects**.
If we allow repeated colours, for instance by reintroducing in the box a ball of the same colour as the one just extracted, each time we choose among n. After $k > 0$ choices there are then n^k possible sequences of colours: n^k is the number of **permutations of n objects in sequences of k, with repetitions** (i.e., allowing an object to be chosen more than once). □

Given two natural numbers n and k such that $0 \le k \le n$, one calls **binomial coefficient** the number

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.10}$$

(the symbol $\binom{n}{k}$ is usually read 'n choose k'). Notice that if $0 < k < n$

$$n! = 1\cdot\ldots\cdot n = 1\cdot\ldots\cdot(n-k)(n-k+1)\cdot\ldots\cdot(n-1)n = (n-k)!(n-k+1)\cdot\ldots\cdot(n-1)n,$$

so simplifying and rearranging the order of factors at the numerator, (1.10) becomes

$$\binom{n}{k} = \frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}, \tag{1.11}$$

another common expression for the binomial coefficient. From definition (1.10) it follows directly that

$$\binom{n}{k} = \binom{n}{n-k}$$

and

$$\binom{n}{0} = \binom{n}{n} = 1, \qquad \binom{n}{1} = \binom{n}{n-1} = n.$$

Moreover, it is easy to prove that for any $n \ge 1$ and any k with $0 < k < n$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \tag{1.12}$$

which provides a convenient means for computing binomial coefficients *recursively*; the coefficients relative to n objects are easily determined once those involving $n-1$ objects are computed. The same formula suggests to write down binomial coefficients in a triangular pattern, known as *Pascal's triangle*[1] (Fig. 1.7): each coefficient of a given row, except for the 1's on the boundary, is the sum of the two numbers that lie above it in the preceding row, precisely as (1.12) prescribes. The construction of Pascal's triangle shows that the binomial coefficients are natural numbers.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1\ldots & & & & & & & & & & \ldots 1 \end{array}$$

Figure 1.7. Pascal's triangle

The term 'binomial coefficient' originates from the power expansion of the polynomial $a+b$ in terms of powers of a and b. The reader will remember the important identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{and} \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The coefficients showing up are precisely the binomial coefficients for $n = 2$ and $n = 3$. In general, for any $n \geq 0$, the formula

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \ldots + \binom{n}{k} a^{n-k}b^k + \ldots + nab^{n-1} + b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k\end{aligned} \tag{1.13}$$

holds, known as **(Newton's) binomial expansion**. This formula is proven with (1.12) using a *proof by induction* (see Appendix A.1, p. 428).

Example 1.9 (continuation)

Given n balls of different colours, let us fix k with $0 \leq k \leq n$. How many different sets of k balls can we form?

Extracting one ball at a time for k times, we already know that there are $n(n-1)\ldots(n-k+1)$ outcomes. On the other hand the same k balls, extracted in a different order, will yield the same set. Since the possible orderings of k elements are $k!$, we see that the number of distinct sets of k balls chosen from n is $\dfrac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!} = \dbinom{n}{k}$. This coefficient represents the number of **combinations of n objects taken k at a time**. Equivalently, the number of subsets of k elements of a set of cardinality n.

[1] Sometimes the denomination *Tartaglia's triangle* appears.

Formula (1.13) with $a = b = 1$ shows that the sum of all binomial coefficients with n fixed equals 2^n, non-incidentally also the total number of subsets of a set with n elements. □

1.5 Cartesian product

Let X, Y be non-empty sets. Given elements x in X and y in Y, we construct the **ordered pair** of numbers

$$(x, y),$$

whose *first component* is x and *second component* is y. An ordered pair is conceptually other than a set of two elements. As the name says, in an ordered pair the order of the components is paramount. This is not the case for a set. If $x \neq y$ the ordered pairs (x, y) and (y, x) are distinct, while $\{x, y\}$ and $\{y, x\}$ coincide as sets.

The set of all ordered pairs (x, y) when x varies in X and y varies in Y is the **Cartesian product** of X and Y, which is indicated by $X \times Y$. Mathematically,

$$\boxed{X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.}$$

The Cartesian product is represented using a rectangle, whose basis corresponds to the set X and whose height is Y (as in Fig. 1.8).

If the sets X, Y are different, the product $X \times Y$ will not be equal to $Y \times X$, in other words the Cartesian product is not commutative.

But if $Y = X$, it is customary to put $X \times X = X^2$ for brevity. In this case the subset of X^2

$$\Delta = \{(x, y) \in X^2 \mid x = y\}$$

of pairs with equal components is called the *diagonal* of the Cartesian product.

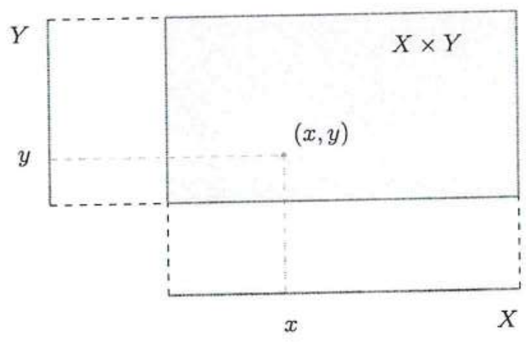

Figure 1.8. Cartesian product of sets

The most significant example of Cartesian product stems from $X = Y = \mathbb{R}$. The set $\mathbb{R}^2$ consists of ordered pairs of real numbers. Just as the set $\mathbb{R}$ mathematically represents a straight line, so $\mathbb{R}^2$ *is a model of the plane* (Fig. 1.9, left). In order to define this correspondence, choose a straight line in the plane and fix on it an origin O, a positive direction and a length unit. This shall be the *x-axis*. Rotating this line counter-clockwise around the origin by $90°$ generates the *y-axis*. In this way we have now an orthonormal *frame* (we only mention that it is sometimes useful to consider frames whose axes are not orthogonal, and/or the units on the axes are different).

Given any point P on the plane, let us draw the straight lines parallel to the axes passing through the point. Denote by x the real number corresponding to the intersection of the x-axis with the parallel to the y-axis, and by y the real number corresponding to the intersection of the y-axis with the parallel to the x-axis. An ordered pair $(x, y) \in \mathbb{R}^2$ is thus associated to each point P on the plane, and vice versa. The components of the pair are called *(Cartesian) coordinates* of P in the chosen frame.

The notion of Cartesian product can be generalised to the product of more sets. Given n non-empty sets $X_1, X_2, \ldots, X_n$, one considers ordered n–tuples

$$(x_1, x_2, \ldots, x_n)$$

where, for every $i = 1, 2, \ldots, n$, each component x_i lives in the set X_i. The Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ is then the set of all such n–tuples.

When $X_1 = X_2 = \ldots = X_n = X$ one simply writes $X \times X \times \ldots \times X = X^n$. In particular, $\mathbb{R}^3$ is the set of triples (x, y, z) of real numbers, and *represents a mathematical model of three-dimensional space* (Fig. 1.9, right).

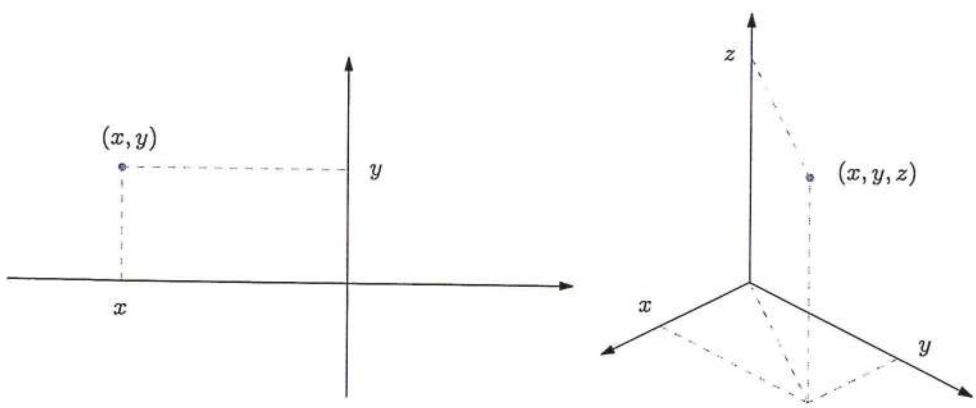

Figure 1.9. Models of the plane (left) and of space (right)

1.6 Relations in the plane

We call *Cartesian plane* a plane equipped with an orthonormal Cartesian frame built as above, which we saw can be identified with the product $\mathbb{R}^2$.

Every non-empty subset R of $\mathbb{R}^2$ defines a **relation** between real numbers; precisely, one says *x is R-related to y*, or *x is related to y by R*, if the ordered pair (x, y) belongs to R. The *graph* of the relation is the set of points in the plane whose coordinates belong to R.

A relation is commonly defined by one or more (in)equalities involving the variables x and y. The subset R is then defined as the set of pairs (x, y) such that x and y satisfy the constraints. Finding R often means determining its graph in the plane. Let us see some examples.

Examples 1.10

i) An equation like

$$ax + by = c,$$

with a, b constant and not both vanishing, defines a straight line. If $b = 0$, the line is parallel to the y-axis, whereas $a = 0$ yields a parallel to the x-axis. Assuming $b \neq 0$ we can write the equation as

$$y = mx + q,$$

where $m = -\frac{a}{b}$ and $q = \frac{c}{b}$. The number m is called *slope* of the line. The line can be plotted by finding the coordinates of two points that belong to it, hence two distinct pairs (x, y) solving the equation. In particular $c = 0$ (or $q = 0$) if and only if the origin belongs to the line. The equation $x - y = 0$ for example defines the bisectrix of the first and third quadrants of the plane.

ii) Replacing the '$=$' sign by '$<$' above, consider the inequality

$$ax + by < c.$$

It defines one of the half-planes in which the straight line of equation $ax+by = c$ divides the plane (Fig. 1.10). If $b > 0$ for instance, the half-plane below the line is obtained. This set is open, i.e., it does not contain the straight line, since the inequality is strict. The inequality $ax + by \leq c$ defines instead a closed set, i.e., including the line.

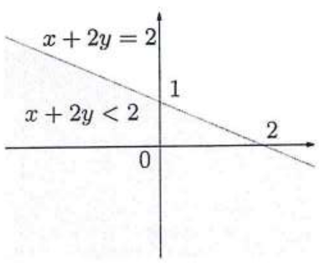

Figure 1.10. Graph of the relation of Example 1.10 ii)

iii) The system

$$\begin{cases} y > 0, \\ x - y \geq 0, \end{cases}$$

defines the intersection between the open half-plane above the x-axis and the closed half-plane lying below the bisectrix of the first and third quadrants. Thus the system describes (Fig. 1.11, left) the wedge between the positive x-axis and the bisectrix (the points on the x-axis are excluded).

iv) The inequality

$$|x - y| < 2$$

is equivalent, recall (1.2), to

$$-2 < x - y < 2.$$

The inequality on the left is in turn equivalent to $y < x+2$, so it defines the open half-plane below the line $y = x + 2$; similarly, the inequality on the right is the same as $y > x-2$ and defines the open half-plane above the line $y = x-2$. What we get is therefore the strip between the two lines, these excluded (Fig. 1.11, right).

v) By Pythagoras's Theorem, the equation

$$x^2 + y^2 = 1$$

defines the set of points P in the plane with distance 1 from the origin of the axes, that is, the circle centred at the origin with radius 1 (in trigonometry it goes under the name of *unit circle*). The inequality

$$x^2 + y^2 \leq 1$$

then defines the disc bounded by the unit circle (Fig. 1.12, left).

vi) The equation

$$y = x^2$$

yields the parabola with vertical axis, vertex at the origin and passing through the point P of coordinates $(1, 1)$.

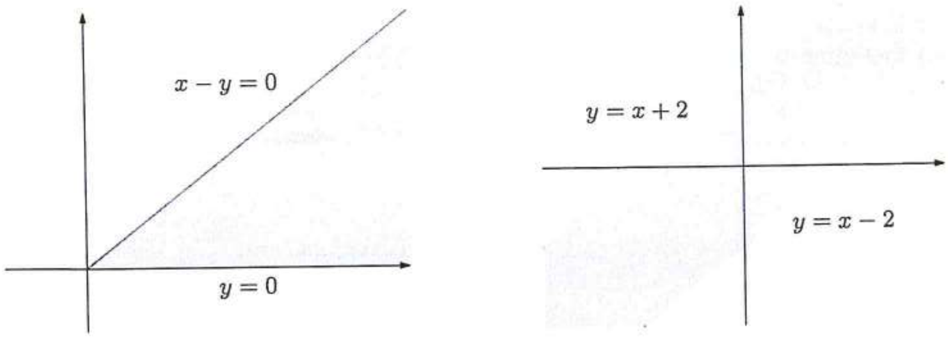

Figure 1.11. Graphs of the relations of Examples 1.10 iii) (left) and 1.10 iv) (right)

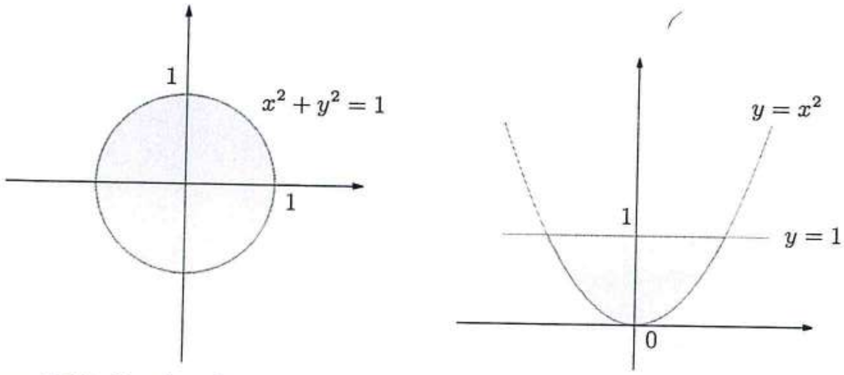

Figure 1.12. Graphs of the relations in Examples 1.10 v) (left) and 1.10 vi) (right)

Thus the inequalities

$$x^2 \le y \le 1$$

define the region enclosed by the parabola and by the straight line given by $y = 1$ (Fig. 1.12, right). □

1.7 Exercises

1. *Solve the following inequalities:*

a) $\dfrac{2x-1}{x-3} > 0$

b) $\dfrac{1-7x}{3x+5} > 0$

c) $\dfrac{x-1}{x-2} > \dfrac{2x-3}{x-3}$

d) $\dfrac{|x|}{x-1} > \dfrac{x+1}{2x-1}$

e) $\dfrac{2x+3}{x+5} \le \dfrac{x+1}{|x-1|}$

f) $\sqrt{x^2-6x} > x+2$

g) $x - 3 \le \sqrt{x^2-2x}$

h) $\dfrac{x+3}{(x+1)^2\sqrt{x^2-3}} \ge 0$

i) $\sqrt{|x^2-4|} - x \ge 0$

ℓ) $\dfrac{x\sqrt{|x^2-4|}}{x^2-4} - 1 > 0$

2. *Describe the following subsets of* $\mathbb{R}$*:*

a) $A = \{x \in \mathbb{R} : x^2 + 4x + 13 < 0\} \cap \{x \in \mathbb{R} : 3x^2 + 5 > 0\}$

b) $B = \{x \in \mathbb{R} : (x+2)(x-1)(x-5) < 0\} \cap \{x \in \mathbb{R} : \dfrac{3x+1}{x-2} \ge 0\}$

c) $C = \{x \in \mathbb{R} : \dfrac{x^2-5x+4}{x^2-9} < 0\} \cup \{x \in \mathbb{R} : \sqrt{7x+1} + x = 17\}$

d) $D = \{x \in \mathbb{R} : x - 4 \ge \sqrt{x^2-6x+5}\} \cup \{x \in \mathbb{R} : x + 2 > \sqrt{x-1}\}$

3. *Determine and draw a picture of the following subsets of* $\mathbb{R}^2$*:*

a) $A = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ b) $B = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 > 0\}$

c) $C = \{(x, y) \in \mathbb{R}^2 : |y - x^2| < 1\}$ d) $D = \{(x, y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} \geq 1\}$

e) $E = \{(x, y) \in \mathbb{R}^2 : 1 + xy > 0\}$ f) $F = \{(x, y) \in \mathbb{R}^2 : x - y \neq 0\}$

4. *Tell whether the following subsets of* $\mathbb{R}$ *are bounded from above and/or below, specifying upper and lower bounds, plus maximum and minimum (if existent):*

a) $A = \{x \in \mathbb{R} : x = n \text{ or } x = \frac{1}{n^2},\ n \in \mathbb{N} \setminus \{0\}\}$

b) $B = \{x \in \mathbb{R} : -1 < x \leq 1 \text{ or } x = 20\}$

c) $C = \{x \in \mathbb{R} : 0 \leq x < 1 \text{ or } x = \frac{2n-3}{n-1},\ n \in \mathbb{N} \setminus \{0, 1\}\}$

d) $D = \{z \in \mathbb{R} : z = xy \text{ with } x, y \in \mathbb{R},\ -1 \leq x \leq 2,\ -3 \leq y < -1\}$

1.7.1 Solutions

1. *Inequalities:*

a) This is a fractional inequality. A fraction is positive if and only if numerator and denominator have the same sign. As $N(x) = 2x - 1 > 0$ if $x > 1/2$, and $D(x) = x - 3 > 0$ for $x > 3$, the inequality holds when $x < 1/2$ or $x > 3$.

b) $-\frac{5}{3} < x < \frac{1}{7}$.

c) Shift all terms to the left and simplify:

$$\frac{x-1}{x-2} - \frac{2x-3}{x-3} > 0, \qquad \text{i.e.,} \qquad \frac{-x^2+3x-3}{(x-2)(x-3)} > 0.$$

The roots of the numerator are not real, so $N(x) < 0$ always. The inequality thus holds when $D(x) < 0$, hence $2 < x < 3$.

d) Moving terms to one side and simplifying yields:

$$\frac{|x|}{x-1} - \frac{x+1}{2x-1} > 0, \qquad \text{i.e.,} \qquad \frac{|x|(2x-1) - x^2 + 1}{(x-1)(2x-1)} > 0.$$

Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$, we study the two cases separately.

When $x \geq 0$ the inequality reads

$$\frac{2x^2 - x - x^2 + 1}{(x-1)(2x-1)} > 0, \qquad \text{or} \qquad \frac{x^2 - x + 1}{(x-1)(2x-1)} > 0.$$

The numerator has no real roots, hence $x^2 - x + 1 > 0$ for all x. Therefore the inequality is satisfied if the denominator is positive. Taking the constrain $x \geq 0$ into account, this means $0 \leq x < 1/2$ or $x > 1$.
When $x < 0$ we have

$$\frac{-2x^2 + x - x^2 + 1}{(x-1)(2x-1)} > 0\,, \qquad \text{i.e.,} \qquad \frac{-3x^2 + x + 1}{(x-1)(2x-1)} > 0\,.$$

$N(x)$ is annihilated by $x_1 = \frac{1-\sqrt{13}}{6}$ and $x_2 = \frac{1+\sqrt{13}}{6}$, so $N(x) > 0$ for $x_1 < x < x_2$ (notice that $x_1 < 0$ and $x_2 \in (\frac{1}{2}, 1)$). As above the denominator is positive when $x < 1/2$ and $x > 1$. Keeping $x < 0$ in mind, we have $x_1 < x < 0$.
The initial inequality is therefore satisfied by any $x \in (x_1, \frac{1}{2}) \cup (1, +\infty)$.

e) $-5 < x \leq -2$, $-\frac{1}{3} \leq x < 1$, $1 < x \leq \frac{5+\sqrt{57}}{2}$; f) $x < -\frac{2}{5}$.

g) First of all observe that the right-hand side is always ≥ 0 where defined, hence when $x^2 - 2x \geq 0$, i.e., $x \leq 0$ or $x \geq 2$. The inequality is certainly true if the left-hand side $x - 3$ is ≤ 0, so for $x \leq 3$.
If $x - 3 > 0$, we take squares to obtain

$$x^2 - 6x + 9 \leq x^2 - 2x\,, \qquad \text{i.e.,} \qquad 4x \geq 9\,, \qquad \text{whence} \qquad x \geq \frac{9}{4}\,.$$

Gathering all information we conclude that the starting inequality holds wherever it is defined, that is for $x \leq 0$ and $x \geq 2$.

h) $x \in [-3, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$.

i) As $|x^2 - 4| \geq 0$, $\sqrt{|x^2-4|}$ is well defined. Let us write the inequality in the form

$$\sqrt{|x^2-4|} \geq x\,.$$

If $x \leq 0$ the inequality is always true, for the left-hand side is positive. If $x > 0$ we square:

$$|x^2 - 4| \geq x^2\,.$$

Note that

$$|x^2-4| = \begin{cases} x^2 - 4 & \text{if } x \leq -2 \text{ or } x \geq 2, \\ -x^2 + 4 & \text{if } -2 < x < 2\,. \end{cases}$$

Consider the case $x \geq 2$ first; the inequality becomes $x^2 - 4 \geq x^2$, which is never true.
Let now $0 < x < 2$; then $-x^2 + 4 \geq x^2$, hence $x^2 - 2 \leq 0$. Thus $0 < x \leq \sqrt{2}$ must hold.
In conclusion, the inequality holds for $x \leq \sqrt{2}$.

ℓ) $x \in (-2, -\sqrt{2}) \cup (2, +\infty)$.

2. *Subsets of* $\mathbb{R}$*:*

a) Because $x^2 + 4x + 13 = 0$ cannot be solved over the reals, the condition $x^2 + 4x + 13 < 0$ is never satisfied and the first set is empty. On the other hand, $3x^2 + 5 > 0$ holds for every $x \in \mathbb{R}$, therefore the second set is the whole $\mathbb{R}$. Thus $A = \emptyset \cap \mathbb{R} = \emptyset$.

b) $B = (-\infty, -2) \cup (2, 5)$.

c) We can write

$$\frac{x^2 - 5x + 4}{x^2 - 9} = \frac{(x-4)(x-1)}{(x-3)(x+3)},$$

whence the first set is $(-3, 1) \cup (3, 4)$.
To find the second set, let us solve the irrational equation $\sqrt{7x+1} + x = 17$, which we write as $\sqrt{7x+1} = 17 - x$. The radicand must necessarily be positive, hence $x \geq -\frac{1}{7}$. Moreover, a square root is always ≥ 0, so we must impose $17 - x \geq 0$, i.e., $x \leq 17$. Thus for $-\frac{1}{7} \leq x \leq 17$, squaring yields

$$7x + 1 = (17 - x)^2, \qquad x^2 - 41x + 288 = 0.$$

The latter equation has two solutions $x_1 = 9$, $x_2 = 32$ (which fails the constraint $x \leq 17$, and as such cannot be considered). The second set then contains only $x = 9$.
Therefore $C = (-3, 1) \cup (3, 4) \cup \{9\}$.

d) $D = [1, +\infty)$.

3. *Subsets of* $\mathbb{R}^2$:

a) The condition holds if x and y have equal signs, thus in the first and third quadrants including the axes (Fig. 1.13, left).

b) See Fig. 1.13, right.

c) We have

$$|y - x^2| = \begin{cases} y - x^2 & \text{if } y \geq x^2, \\ x^2 - y & \text{if } y \leq x^2. \end{cases}$$

Demanding $y \geq x^2$ means looking at the region in the plane bounded from below by the parabola $y = x^2$. There, we must have

$$y - x^2 < 1, \qquad \text{i.e.,} \qquad y < x^2 + 1,$$

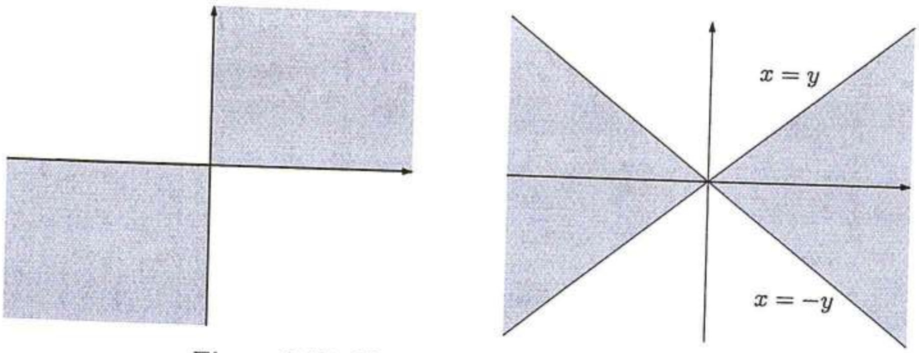

Figure 1.13. The sets A and B of Exercise 3

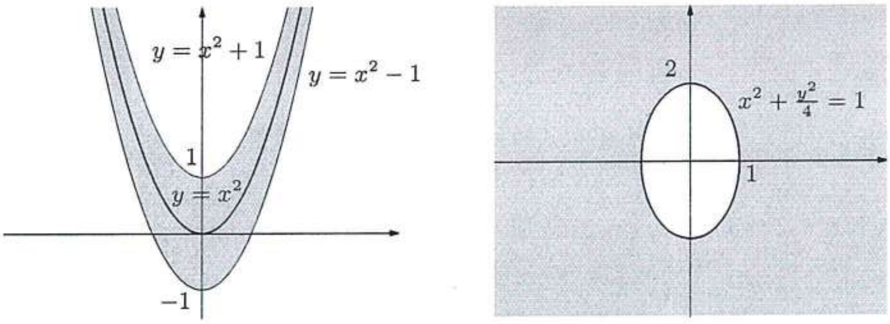

Figure 1.14. The sets C and D of Exercise 3

that is $x^2 \le y < x^2 + 1$.
Vice versa if $y < x^2$,

$$x^2 - y < 1\,, \qquad \text{i.e.,} \qquad y > x^2 - 1\,,$$

hence $x^2 - 1 < y \le x^2$.
Eventually, the required region is confined by (though does not include) the parabolas $y = x^2 - 1$ and $y = x^2 + 1$ (Fig. 1.14, left).

d) See Fig. 1.14, right.

e) For $x > 0$ the condition $1 + xy > 0$ is the same as $y > -\frac{1}{x}$. Thus we consider all points of the first and third quadrants above the hyperbola $y = -\frac{1}{x}$.
For $x < 0$, $1 + xy > 0$ means $y < -\frac{1}{x}$, satisfied by the points in the second and fourth quadrants this time, lying below the hyperbola $y = -\frac{1}{x}$.
At last, if $x = 0$, $1 + xy > 0$ holds for any y, implying that the y-axis belongs to the set E.
Therefore: the region lies between the two branches of the hyperbola (these are not part of E) $y = -\frac{1}{x}$, including the y-axis (Fig. 1.15, left).

f) See Fig. 1.15, right.

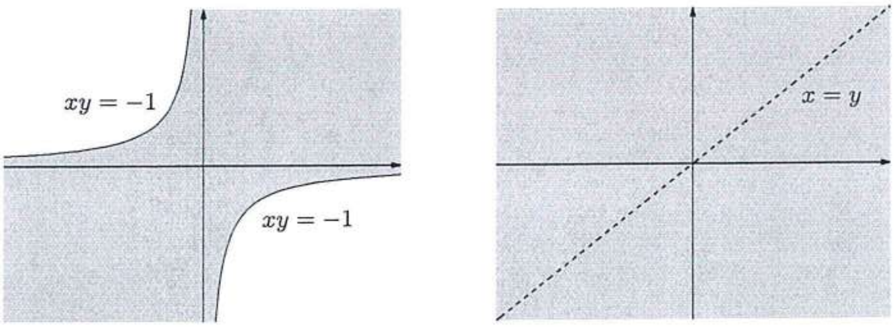

Figure 1.15. The sets E and F of Exercise 3

4. *Bounded and unbounded sets:*

a) We have $A = \{1, 2, 3, \ldots, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\}$. Since $\mathbb{N} \setminus \{0\} \subset A$, the set A is not bounded from above, hence $\sup A = +\infty$ and there is no maximum.
In addition, the fact that every element of A is positive makes A bounded from below. We claim that 0 is the greatest lower bound of A. In fact, if $r > 0$ were a lower bound of A, then $\frac{1}{n^2} > r$ for any non-zero $n \in \mathbb{N}$. This is the same as $n^2 < \frac{1}{r}$, hence $n < \frac{1}{\sqrt{r}}$. But the last inequality is absurd since natural numbers are not bounded from above. Finally $0 \notin A$, so we conclude $\inf A = 0$ and A has no minimum.

b) $\inf B = -1$, $\sup B = \max B = 20$, and $\min B$ does not exist.

c) $C = [0, 1] \cup \{\frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \ldots\} \subset [0, 2)$; then C is bounded, and $\inf C = \min C = 0$. Since $\dfrac{2n-3}{n-1} = 2 - \dfrac{1}{n-1}$, it is not hard to show that $\sup C = 2$, although there is no maximum in C.

d) $\inf C = \min C = -6$, $\sup B = \max B = 3$.