2

Functions

Functions crop up regularly in everyday life (for instance: each student of the Polytechnic of Turin has a unique identification number), in physics (to each point of a region in space occupied by a fluid we may associate the velocity of the particle passing through that point at a given moment), in economy (each working day at Milan's stock exchange is tagged with the Mibtel index), and so on.

The mathematical notion of a function subsumes all these situations.

2.1 Definitions and first examples

Let X and Y be two sets. A **function** f **defined on** X **with values in** Y is a correspondence associating to each element $x \in X$ *at most* one element $y \in Y$. This is often shortened to 'a function from X to Y'. A synonym for function is **map**. The set of $x \in X$ to which f associates an element in Y is the **domain** of f; the domain is a subset of X, indicated by $\operatorname{dom} f$. One writes

$$\boxed{f : \operatorname{dom} f \subseteq X \to Y.}$$

If $\operatorname{dom} f = X$, one says that f is defined **on** X and writes simply $f : X \to Y$.

The element $y \in Y$ associated to an element $x \in \operatorname{dom} f$ is called the **image of** x **by** or **under** f and denoted $y = f(x)$. Sometimes one writes

$$f : x \mapsto f(x).$$

The set of images $y = f(x)$ of all points in the domain constitutes the **range of** f, a subset of Y indicated by $\operatorname{im} f$.

The **graph** of f is the subset $\Gamma(f)$ of the Cartesian product $X \times Y$ made of pairs $(x, f(x))$ when x varies in the domain of f, i.e.,

$$\boxed{\Gamma(f) = \big\{(x, f(x)) \in X \times Y \ : \ x \in \operatorname{dom} f\big\}.} \tag{2.1}$$

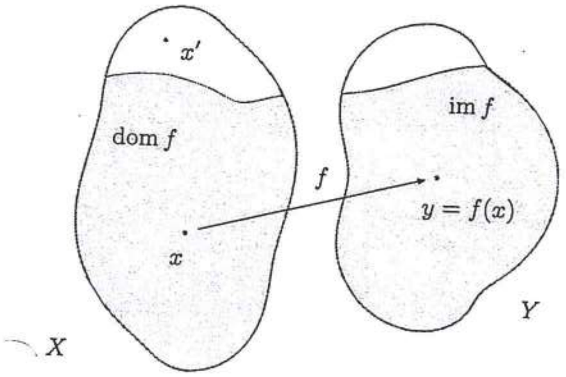

Figure 2.1. Naive representation of a function using Venn diagrams

In the sequel we shall consider maps between sets of numbers most of the time. If $Y = \mathbb{R}$, the function f is said **real** or **real-valued**. If $X = \mathbb{R}$, the function is of **one real variable**. Therefore the graph of a real function is a subset of the Cartesian plane $\mathbb{R}^2$.

A remarkable special case of map arises when $X = \mathbb{N}$ and the domain contains a set of the type $\{n \in \mathbb{N} \, : \, n \geq n_0\}$ for a certain natural number $n_0 \geq 0$. Such a function is called **sequence**. Usually, indicating by a the sequence, it is preferable to denote the image of the natural number n by the symbol a_n rather than $a(n)$; thus we shall write $a : n \mapsto a_n$. A common way to denote sequences is $\{a_n\}_{n \geq n_0}$ (ignoring possible terms with $n < n_0$) or even $\{a_n\}$.

Examples 2.1

Let us consider examples of real functions of real variable.

i) $f : \mathbb{R} \to \mathbb{R}, \; f(x) = ax + b$ (a, b real coefficients), whose graph is a straight line (Fig. 2.2, top left).

ii) $f : \mathbb{R} \to \mathbb{R}, \; f(x) = x^2$, whose graph is a parabola (Fig. 2.2, top right).

iii) $f : \mathbb{R} \backslash \{0\} \subset \mathbb{R} \to \mathbb{R}, \; f(x) = \frac{1}{x}$, has a rectangular hyperbola in the coordinate system of its asymptotes as graph (Fig. 2.2, bottom left).

iv) A real function of a real variable can be defined by multiple expressions on different intervals, in which case is it called a **piecewise function**. An example is given by $f : [0, 3] \to \mathbb{R}$

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1, \\ 4 - x & \text{if } 1 < x \leq 2, \\ x - 1 & \text{if } 2 < x \leq 3, \end{cases} \tag{2.2}$$

drawn in Fig. 2.2, bottom right.

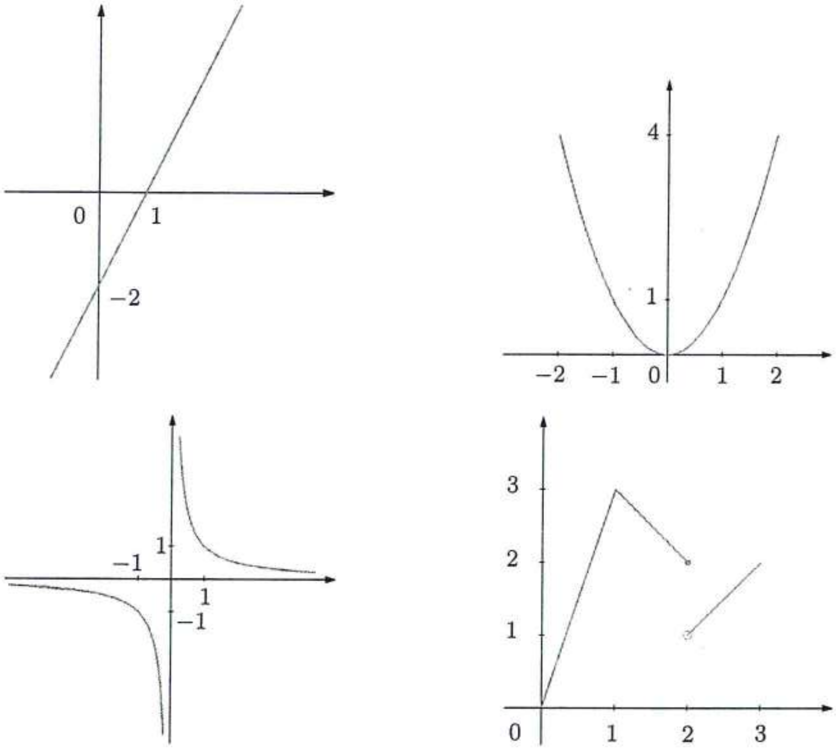

Figure 2.2. Graphs of the maps $f(x) = 2x-2$ (top left), $f(x) = x^2$ (top right), $f(x) = \dfrac{1}{x}$ (bottom left) and of the piecewise function (2.2) (bottom right)

Among piecewise functions, the following are particularly important:

v) the **absolute value** (Fig. 2.3, top left)

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0; \end{cases}$$

vi) the **sign** (Fig. 2.3, top right)

$$f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \operatorname{sign}(x) = \begin{cases} +1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0; \end{cases}$$

vii) the **integer part** (Fig. 2.3, bottom left), also known as **floor function**,

$$f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = [x] = \text{ the greatest integer } \leq x$$

(for example, $[4] = 4$, $[\sqrt{2}] = 1$, $[-1] = -1$, $[-\frac{3}{2}] = -2$); notice that

$$[x] \leq x < [x] + 1, \qquad \forall x \in \mathbb{R};$$

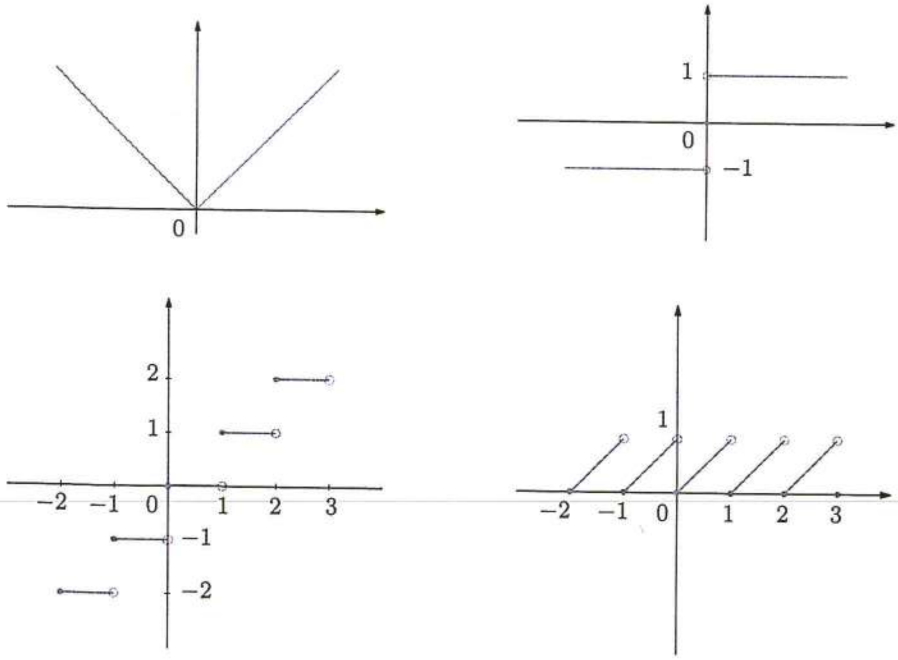

Figure 2.3. Clockwise from top left: graphs of the functions: absolute value, sign, mantissa and integer part

viii) the **mantissa** (Fig. 2.3, bottom right)

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = M(x) = x - [x]$$

(the property of the floor function implies $0 \leq M(x) < 1$).

Let us give some examples of sequences now.

ix) The sequence

$$a_n = \frac{n}{n+1} \tag{2.3}$$

is defined for all $n \geq 0$. The first few terms read

$$a_0 = 0\,, \quad a_1 = \frac{1}{2} = 0.5\,, \quad a_2 = \frac{2}{3} = 0.\overline{6}\,, \quad a_3 = \frac{3}{4} = 0.75\,.$$

Its graph is shown in Fig. 2.4 (top left).

x) The sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n \tag{2.4}$$

is defined for $n \geq 1$. The first terms are

$$a_1 = 2\,, \quad a_2 = \frac{9}{4} = 2.25\,, \quad a_3 = \frac{64}{27} = 2.37\overline{037}\,, \quad a_4 = \frac{625}{256} = 2.44140625\,.$$

Fig. 2.4 (top right) shows the graph of such sequence.

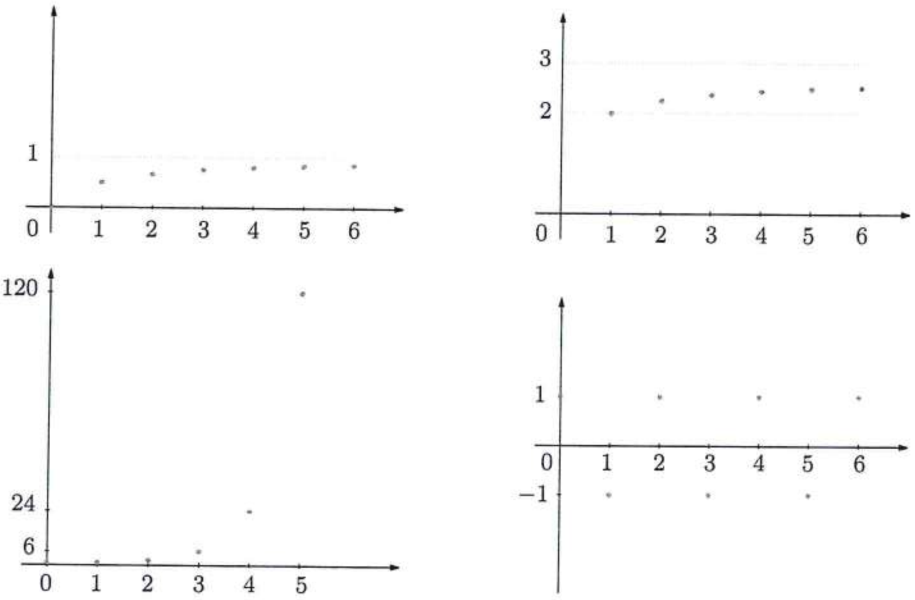

Figure 2.4. Clockwise: graphs of the sequences (2.3), (2.4), (2.6), (2.5)

xi) The sequence

$$a_n = n! \tag{2.5}$$

associates to each natural number its factorial, defined in (1.9). The graph of this sequence is shown in Fig. 2.4 (bottom left); as the values of the sequence grow rapidly as n increases, we used different scalings on the coordinate axes.

xii) The sequence

$$a_n = (-1)^n = \begin{cases} +1 & \text{if } n \text{ is even}, \\ -1 & \text{if } n \text{ is odd}, \end{cases} \qquad (n \geq 0) \tag{2.6}$$

has alternating values $+1$ and -1, according to the parity of n. The graph of the sequence is shown in Fig. 2.4 (bottom right).

At last, here are two maps defined on $\mathbb{R}^2$ (functions of *two real variables*).

xiii) The function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \sqrt{x^2 + y^2}$$

maps a generic point P of the plane with coordinates (x, y) to its distance from the origin.

xiv) The map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (y, x)$$

associates to a point P the point P' symmetric to P with respect to the bisectrix of the first and third quadrants. □

Consider a map from X to Y. One should take care in noting that the symbol for an element of X (to which one refers as the *independent variable*) and the symbol for an element in Y (*dependent variable*), are completely arbitary. What really determines the function is the way of associating each element of the domain to its corresponding image. For example, if x, y, z, t are symbols for real numbers, the expressions $y = f(x) = 3x$, $x = f(y) = 3y$, or $z = f(t) = 3t$ denote the *same* function, namely the one mapping each real number to its triple.

2.2 Range and pre-image

Let A be a subset of X. The **image of A under f** is the set

$$f(A) = \{f(x) \ : \ x \in A\} \subseteq \operatorname{im} f$$

of all the images of elements of A. Notice that $f(A)$ is empty if and only if A contains no elements of the domain of f. The image $f(X)$ of the whole set X is the range of f, already denoted by $\operatorname{im} f$.

Let y be any element of Y; the **pre-image of y by f** is the set

$$f^{-1}(y) = \{x \in \operatorname{dom} f \ : \ f(x) = y\}$$

of elements in X whose image is y. This set is empty precisely when y does not belong to the range of f. If B is a subset of Y, the **pre-image of B under f** is defined as the set

$$f^{-1}(B) = \{x \in \operatorname{dom} f \ : \ f(x) \in B\},$$

union of all pre-images of elements of B.

It is easy to check that $A \subseteq f^{-1}(f(A))$ for any subset A of $\operatorname{dom} f$, and $f(f^{-1}(B)) = B \cap \operatorname{im} f \subseteq B$ for any subset B of Y.

Example 2.2

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. The image under f of the interval $A = [1, 2]$ is the interval $B = [1, 4]$. Yet the pre-image of B under f is the union of the intervals $[-2, -1]$ and $[1, 2]$, namely, the set

$$f^{-1}(B) = \{x \in \mathbb{R} \ : \ 1 \leq |x| \leq 2\}$$

(see Fig. 2.5). □

The notions of infimum, supremum, maximum and minimum, introduced in Sect. 1.3.1, specialise in the case of images of functions.

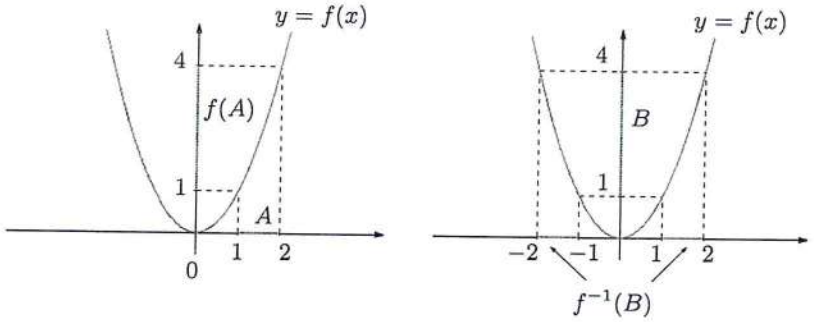

Figure 2.5. Image (left) and pre-image (right) of an interval relative to the function $f(x) = x^2$

> **Definition 2.3** *Let f be a real map and A a subset of* $\operatorname{dom} f$. *One calls* **supremum of f on A** (*or* **in** A) *the supremum of the image of A under f*
>
> $$\sup_{x\in A} f(x) = \sup f(A) = \sup\{f(x) \mid x \in A\}.$$
>
> *Then f is* **bounded from above on** *A if the set $f(A)$ is bounded from above, or equivalently, if $\sup\limits_{x\in A} f(x) < +\infty$.*
> *If $\sup\limits_{x\in A} f(x)$ is finite and belongs to $f(A)$, then it is the maximum of this set. This number is the* **maximum value** (*or simply, the* **maximum**) **of f on** *A and is denoted by $\max\limits_{x\in A} f(x)$.*
> *The concepts of* **infimum** *and of* **minimum** *of f on A are defined similarly. Eventually, f is said* **bounded on** *A if the set $f(A)$ is bounded.*

At times, the shorthand notations $\sup_A f$, $\max_A f$, et c. are used.
The maximum value $M = \max_A f$ of f on the set A is characterised by the conditions:

i) M is a value assumed by the function on A, i.e.,

$$\text{there exists } x_M \in A \text{ such that } f(x_M) = M;$$

ii) M is greater or equal than any other value of the map on A, so

$$\text{for any } x \in A,\ f(x) \le M.$$

Example 2.4

Consider the function $f(x)$ defined in (2.2). One verifies easily

$$\max_{x\in[0,2]} f(x) = 3, \quad \min_{x\in[0,2]} f(x) = 0, \quad \max_{x\in[1,3]} f(x) = 3, \quad \inf_{x\in[1,3]} f(x) = 1.$$

The map does not assume the value 1 anywhere in the interval $[1,3]$, so there is no minimum on that set. □

2.3 Surjective and injective functions; inverse function

A map with values in Y is called **onto** if $\operatorname{im} f = Y$. This means that each $y \in Y$ is the image of one element $x \in X$ at least. The term **surjective (on** Y**)** has the same meaning. For instance, $f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b$ with $a \neq 0$ is surjective on $\mathbb{R}$, or onto: the real number y is the image of $x = \frac{y-b}{a}$. On the contrary, the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is not onto, because its range coincides with the interval $[0, +\infty)$.

A function f is called **one-to-one** (or **1-1**) if every $y \in \operatorname{im} f$ is the image of a unique element $x \in \operatorname{dom} f$. Otherwise put, if $y = f(x_1) = f(x_2)$ for some elements x_1, x_2 in the domain of f, then necessarily $x_1 = x_2$. This, in turn, is equivalent to

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$$

for all $x_1, x_2 \in \operatorname{dom} f$ (see Fig. 2.6). Again, the term **injective** may be used. If a map f is one-to-one, we can associate to each element y in the range the unique x in the domain with $f(x) = y$. Such correspondence determines a function defined on Y and with values in X, called **inverse function** of f and denoted by the symbol f^{-1}. Thus

$$x = f^{-1}(y) \quad \Longleftrightarrow \quad y = f(x)$$

(the notation mixes up deliberately the pre-image of y under f with the unique element this set contains). The inverse function f^{-1} has the image of f as its domain, and the domain of f as range:

$$\operatorname{dom} f^{-1} = \operatorname{im} f, \qquad \operatorname{im} f^{-1} = \operatorname{dom} f.$$

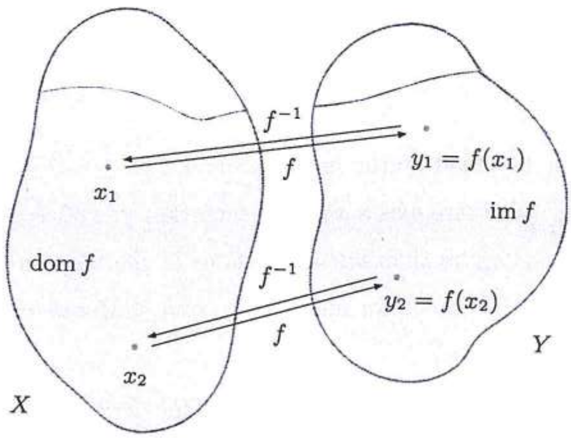

Figure 2.6. Representation of a one-to-one function and its inverse

A one-to-one map is therefore **invertible**; the two notions (injectivity and invertibility) coincide.

What is the link between the graphs of f, defined in (2.1), and of the inverse function f^{-1}? One has

$$\begin{aligned}\Gamma(f^{-1}) &= \{(y, f^{-1}(y)) \in Y \times X \ : \ y \in \operatorname{dom} f^{-1}\} \\ &= \{(f(x), x) \in Y \times X \ : \ x \in \operatorname{dom} f\}.\end{aligned}$$

Therefore, the graph of the inverse map may be obtained from the graph of f by *swapping* the components in each pair. For real functions of one real variable, this corresponds to a reflection in the Cartesian plane with respect to the bisectrix $y = x$ (see Fig. 2.7: a) is reflected into b)). On the other hand, finding the explicit expression $x = f^{-1}(y)$ of the inverse function could be hard, if possible at all.

Provided that the inverse map in the form $x = f^{-1}(y)$ can be determined, often one prefers to denote the independent variable (of f^{-1}) by x, and the dependent variable by y, thus obtaining the expression $y = f^{-1}(x)$. This is merely a change of notation (see the remark at the end of Sect. 2.1). The procedure allows to draw the graph of the inverse function in the same frame system of f (see Fig. 2.7, from b) to c)).

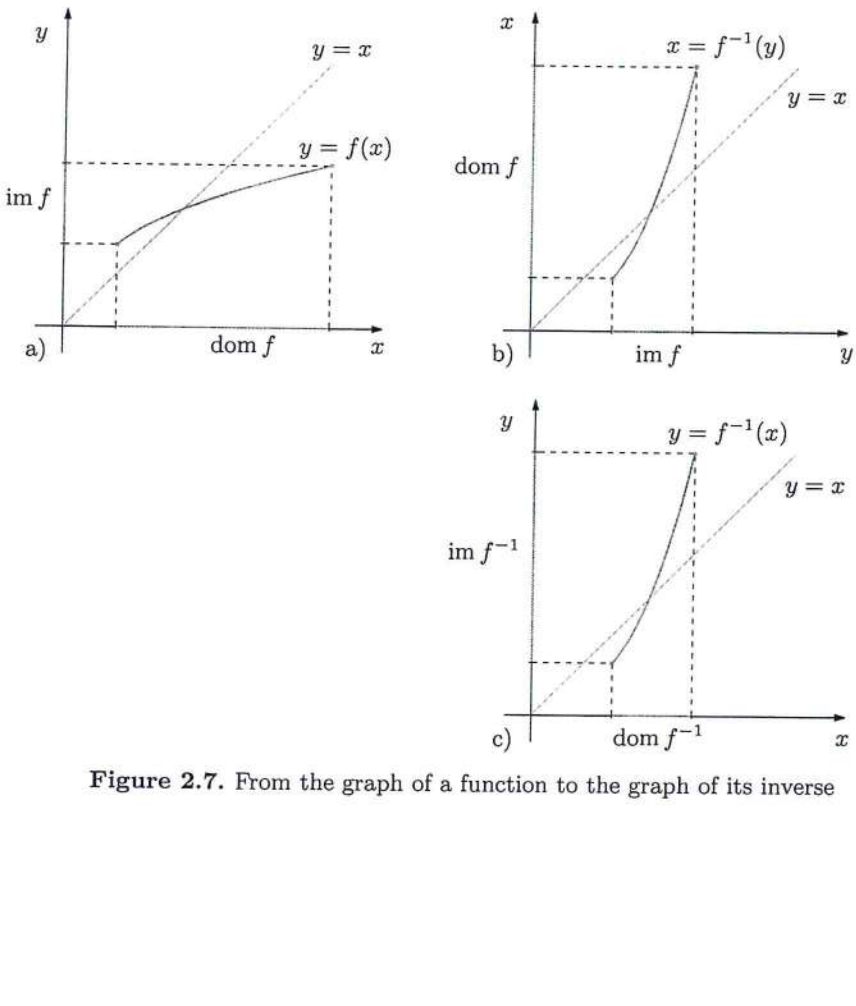

Figure 2.7. From the graph of a function to the graph of its inverse

Examples 2.5

i) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ is one-to-one for all $a \neq 0$ (in fact, $f(x_1) = f(x_2) \Rightarrow ax_1 = ax_2 \Rightarrow x_1 = x_2$). Its inverse is $x = f^{-1}(y) = \frac{y-b}{a}$, or $y = f^{-1}(x) = \frac{x-b}{a}$.

ii) The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not one-to-one because $f(x) = f(-x)$ for any real x. Yet if we consider only values ≥ 0 for the independent variable, i.e., if we **restrict** f to the interval $[0, +\infty)$, then the function becomes 1-1 (in fact, $f(x_1) = f(x_2) \Rightarrow x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = 0 \Rightarrow x_1 = x_2$). The inverse function $x = f^{-1}(y) = \sqrt{y}$ is also defined on $[0, +\infty)$. Conventionally one says that the 'squaring' map $y = x^2$ has the function 'square root' $y = \sqrt{x}$ for inverse (on $[0, +\infty)$). Notice that the restriction of f to the interval $(-\infty, 0]$ is 1-1, too; the inverse in this case is $y = -\sqrt{x}$.

iii) The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is one-to-one. In fact $f(x_1) = f(x_2) \Rightarrow x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \Rightarrow x_1 = x_2$ since $x_1^2 + x_1x_2 + x_2^2 = \frac{1}{2}[x_1^2 + x_2^2 + (x_1 + x_2)^2] > 0$ for any $x_1 \neq x_2$. The inverse function is the 'cubic root' $y = \sqrt[3]{x}$, defined on all $\mathbb{R}$. □

As in Example ii) above, if a function f is not injective over the whole domain, it might be so on a subset $A \subseteq \operatorname{dom} f$. The **restriction of f to A** is the function

$$f_{|A} : A \to Y \qquad \text{such that} \qquad f_{|A}(x) = f(x)\,, \quad \forall x \in A\,,$$

and is therefore invertible.

Let f be defined on X with values Y. If f is one-to-one and onto, it is called a **bijection** (or **bijective function**) from X to Y. If so, the inverse map f^{-1} is defined on Y, and is one-to-one and onto (on X); thus, f^{-1} is a bijection from Y to X.

For example, the functions $f(x) = ax + b$ ($a \neq 0$) and $f(x) = x^3$ are bijections from $\mathbb{R}$ to itself. The function $f(x) = x^2$ is a bijection on $[0, +\infty)$ (i.e., from $[0, +\infty)$ to $[0, +\infty)$).

If f is a bijection between X and Y, the sets X and Y are in **bijective corrispondence** through f: each element of X is assigned to one and only one element of Y, and vice versa. The reader should notice that two *finite* sets (i.e., containing a finite number of elements) are in bijective correspondence if and only if they have the same number of elements. On the contrary, an infinite set can correspond bijectively to a proper subset; the function (sequence) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$, for example, establishes a bijection between $\mathbb{N}$ and the subset of even numbers.

To conclude the section, we would like to mention a significant interpretation of the notions of 1-1, onto, and bijective maps just introduced. Both in pure Mathematics and in applications one is frequently interested in solving a problem, or an equation, of the form

$$f(x) = y\,,$$

where f is a suitable function between two sets X and Y. The quantity y represents the *datum* of the problem, while x stands for the *solution* to the problem, or *the unknown* of the equation. For instance, given the real number y, find the real number x solution of the algebraic equation

$$x^3 + x^2 - \sqrt[3]{x} = y.$$

Well, to say that f is an onto function on Y is the same as saying that the problem or equation of concern admits at least one solution for each given y in Y; asking f to be 1-1 is equivalent to saying the solution, if it exists at all, is unique. Eventually, f bijection from X to Y means that for any given y in Y there is one, and only one, solution $x \in X$.

2.4 Monotone functions

Let f be a real map of one real variable, and I the domain of f or an interval contained in the domain. We would like to describe precisely the situation in which the dependent variable increases or decreases as the independent variable grows. Examples are the increase in the pressure of a gas inside a sealed container as we raise its temperature, or the decrease of the level of fuel in the tank as a car proceeds on a highway. We have the following definition.

Definition 2.6 *The function f is* **increasing on** *I if, given elements x_1, x_2 in I with $x_1 < x_2$, one has $f(x_1) \le f(x_2)$; in symbols*

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \le f(x_2). \tag{2.7}$$

The function f is **strictly increasing on** *I if*

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2). \tag{2.8}$$

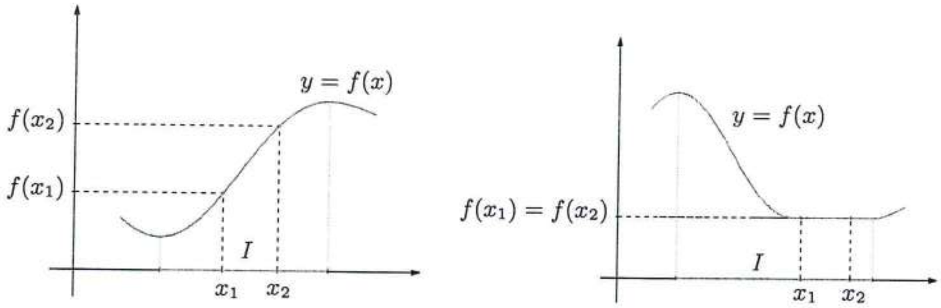

Figure 2.8. Strictly increasing (left) and decreasing (right) functions on an interval I

If a map is strictly increasing then it is increasing as well, hence condition (2.8) is stronger than (2.7).

The definitions of **decreasing** and **strictly decreasing** functions on I are obtained from the previous definitions by reverting the inequality between $f(x_1)$ and $f(x_2)$.

The function f is (**strictly**) **monotone on** I if it is either (strictly) increasing or (strictly) decreasing on I. An interval I where f is monotone is said **interval of monotonicity** of f.

Examples 2.7

i) The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$, is strictly increasing on $\mathbb{R}$ for $a > 0$, constant on $\mathbb{R}$ for $a = 0$ (hence increasing as well as decreasing), and strictly decreasing on $\mathbb{R}$ when $a < 0$.

ii) The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is strictly increasing on $I = [0, +\infty)$. Taking in fact two arbitrary numbers $x_1, x_2 \geq 0$ with $x_1 < x_2$, we have $x_1^2 \leq x_1 x_2 < x_2^2$. Similarly, f is strictly decreasing on $(-\infty, 0]$. It is not difficult to check that all functions of the type $y = x^n$, with $n \geq 4$ even, have the same monotonic behaviour as f (Fig. 2.9, left).

iii) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ strictly increases on $\mathbb{R}$. All functions like $y = x^n$ with n odd have analogous behaviour (Fig. 2.9, right).

iv) Referring to Examples 2.1, the maps $y = [x]$ and $y = \text{sign}(x)$ are increasing (though not strictly increasing) on $\mathbb{R}$.
The mantissa $y = M(x)$ of x, instead, is not monotone on $\mathbb{R}$; but it is nevertheless strictly increasing on each interval $[n, n+1)$, $n \in \mathbb{Z}$. □

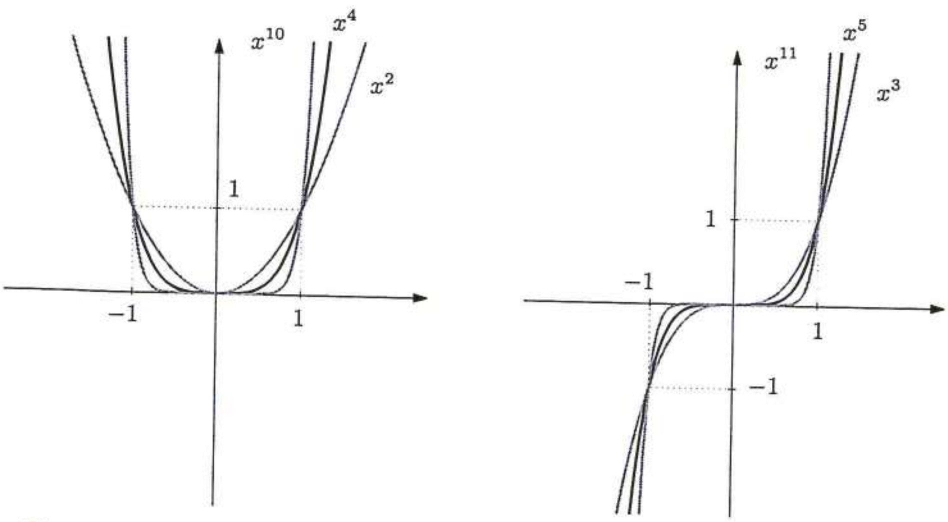

Figure 2.9. Graphs of some functions $y = x^n$ with n even (left) and n odd (right)

Now to a simple yet crucial result.

Proposition 2.8 *If f is strictly monotone on its domain, then f is one-to-one.*

Proof. To fix ideas, let us suppose f is strictly increasing. Given $x_1, x_2 \in \operatorname{dom} f$ with $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_2 < x_1$. In the former case, using (2.8) we obtain $f(x_1) < f(x_2)$, hence $f(x_1) \neq f(x_2)$. In the latter case the same conclusion holds by swapping the roles of x_1 and x_2. □

Under the assumption of the above proposition, there exists the inverse function f^{-1} then; one can comfortably check that f^{-1} is also strictly monotone, and in the same way as f (both are strictly increasing or strictly decreasing). For instance, the strictly increasing function $f : [0, +\infty) \to [0, +\infty)$, $f(x) = x^2$ has, as inverse, the strictly increasing function $f^{-1} : [0, +\infty) \to [0, +\infty)$, $f^{-1}(x) = \sqrt{x}$.

The logic implication

$$f \text{ is strictly monotone on its domain} \qquad \Rightarrow \qquad f \text{ is one-to-one}$$

cannot be reversed. In other words, a map f may be one-to-one without increasing strictly on its domain. For instance $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is one-to-one, actually bijective on $\mathbb{R}$, but it is not strictly increasing, nor strictly decreasing or $\mathbb{R}$. We shall return to this issue in Sect. 4.3.

A useful remark is the following. The sum of functions that are similarly monotone (i.e., all increasing or all decreasing) is still a monotone function of the same kind, and turns out to be strictly monotone if one at least of the summands is. The map $f(x) = x^5 + x$, for instance, is strictly increasing on $\mathbb{R}$, being the sum of two functions with the same property. According to Proposition 2.8 f is then invertible; note however that the relation $f(x) = y$ cannot be made explicit in the form $x = f^{-1}(y)$.

2.5 Composition of functions

Let X, Y, Z be sets. Suppose f is a function from X to Y, and g a function from Y to Z. We can manifacture a new function h from X to Z by setting

$$\boxed{h(x) = g(f(x)).} \tag{2.9}$$

The function h is called **composition of f and g**, sometimes **composite map**, and is indicated by the symbol $h = g \circ f$ (read 'g composed (with) f').

Example 2.9

Consider the two real maps $y = f(x) = x - 3$ and $z = g(y) = y^2 + 1$ of one real variable. The composition of f and g reads $z = h(x) = g \circ f(x) = (x-3)^2 + 1$. □

Bearing in mind definition (2.9), the domain of the composition $g \circ f$ is determined as follows: in order for x to belong to the domain of $g \circ f$, $f(x)$ must be defined, so x must be in the domain of f; moreover, $f(x)$ has to be a element of the domain of g. Thus

$$x \in \operatorname{dom} g \circ f \qquad \Longleftrightarrow \qquad x \in \operatorname{dom} f \quad \text{and} \quad f(x) \in \operatorname{dom} g.$$

The domain of $g \circ f$ is then a *subset* of the domain of f (see Fig. 2.10).

Examples 2.10

i) The domain of $f(x) = \dfrac{x+2}{|x-1|}$ is $\mathbb{R} \setminus \{1\}$, while $g(y) = \sqrt{y}$ is defined on the interval $[0, +\infty)$. The domain of $g \circ f(x) = \sqrt{\dfrac{x+2}{|x-1|}}$ consists of the $x \neq 1$ such that $\dfrac{x+2}{|x-1|} \geq 0$; hence, $\operatorname{dom} g \circ f = [-2, +\infty) \setminus \{1\}$.

ii) Sometimes the composition $g \circ f$ has an empty domain. This happens for instance for $f(x) = \dfrac{1}{1+x^2}$ (notice $f(x) \leq 1$) and $g(y) = \sqrt{y-5}$ (whose domain is $[5, +\infty)$). □

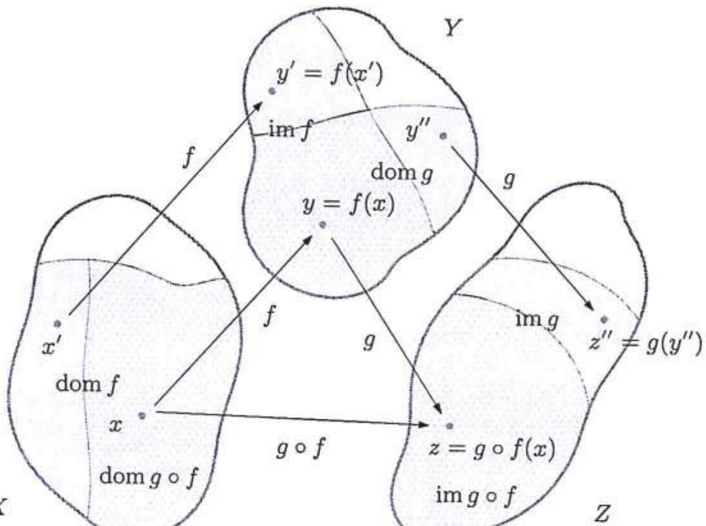

Figure 2.10. Representation of a composite function via Venn diagrams.

The operation of composition is not commutative: if $g \circ f$ and $f \circ g$ are both defined (for instance, when $X = Y = Z$), the two composites do not coincide in general. Take for example $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{1+x}$, for which $g \circ f(x) = \frac{x}{1+x}$, but $f \circ g(x) = 1 + x$.

If f and g are both one-to-one (or both onto, or both bijective), it is not difficult to verify that $g \circ f$ has the same property. In the first case in particular, the formula

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

holds.

Moreover, if f and g are real monotone functions of real variable, $g \circ f$ too will be monotone, or better: $g \circ f$ is increasing if both f and g are either increasing or decreasing, and decreasing otherwise. Let us prove only one of these properties. Let for example f increase and g decrease; if $x_1 < x_2$ are elements in $\operatorname{dom} g \circ f$, the monotone behaviour of f implies $f(x_1) \le f(x_2)$; now the monotonicity of g yields $g(f(x_1)) \ge g(f(x_2))$, so $g \circ f$ is decreasing.

We observe finally that if f is a one-to-one function (and as such it admits inverse f^{-1}), then

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = x, \qquad \forall x \in \operatorname{dom} f,$$
$$f \circ f^{-1}(y) = f(f^{-1}(y)) = y, \qquad \forall y \in \operatorname{im} f.$$

Calling **identity map** on a set X the function $\operatorname{id}_X : X \to X$ such that $\operatorname{id}_X(x) = x$ for all $x \in X$, we have $f^{-1} \circ f = \operatorname{id}_{\operatorname{dom} f}$ and $f \circ f^{-1} = \operatorname{id}_{\operatorname{im} f}$.

2.5.1 Translations, rescalings, reflections

Let f be a real map of one real variable (for instance, the function of Fig. 2.11). Fix a real number $c \neq 0$, and denote by $t_c : \mathbb{R} \to \mathbb{R}$ the function $t_c(x) = x + c$. Composing f with t_c results in a **translation** of the graph of f: precisely, the

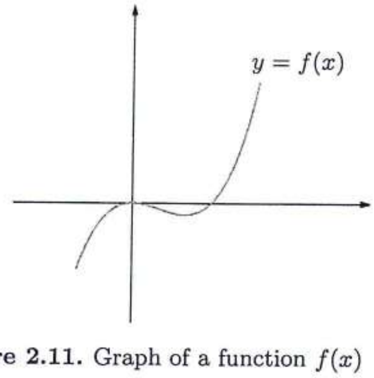

Figure 2.11. Graph of a function $f(x)$

graph of the function $f \circ t_c(x) = f(x+c)$ is shifted horizontally with respect to the graph of f: towards the left if $c > 0$, to the right if $c < 0$. Similarly, the graph of $t_c \circ f(x) = f(x)+c$ is translated vertically with respect to the graph of f, towards the top for $c > 0$, towards the bottom if $c < 0$. Fig. 2.12 provides examples of these situations.

Fix a real number $c > 0$ and denote by $s_c : \mathbb{R} \to \mathbb{R}$ the map $s_c(x) = cx$. The composition of f with s_c has the effect of **rescaling** the graph of f. Precisely, if $c > 1$ the graph of the function $f \circ s_c(x) = f(cx)$ is 'compressed' horizontally towards the y-axis, with respect to the graph of f; if $0 < c < 1$ instead, the graph 'stretches' away from the y-axis. The analogue effect, though in the vertical direction, is seen for the function $s_c \circ f(x) = cf(x)$: here $c > 1$ 'spreads out' the graph away from the x-axis, while $0 < c < 1$ 'squeezes' it towards the axis, see Fig. 2.13.

Notice also that the graph of $f(-x)$ is obtained by **reflecting** the graph of $f(x)$ along the y-axis, like in front of a mirror. The graph of $f(|x|)$ instead coincides with that of f for $x \geq 0$, and for $x < 0$ it is the mirror image of the latter with respect to the vertical axis. At last, the graph of $|f(x)|$ is the same as the graph of f when $f(x) \geq 0$, and is given by reflecting the latter where $f(x) < 0$, see Fig. 2.14.

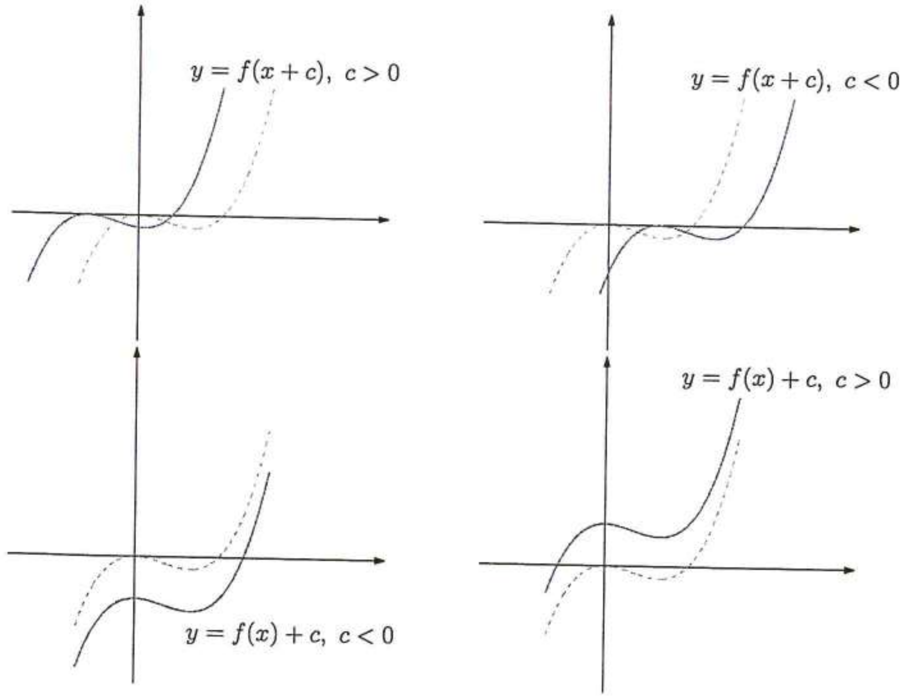

Figure 2.12. Graphs of the functions $f(x+c)$ ($c > 0$: top left, $c < 0$: top right), and $f(x) + c$ ($c < 0$: bottom left, $c > 0$: bottom right), where $f(x)$ is the map of Fig. 2.11

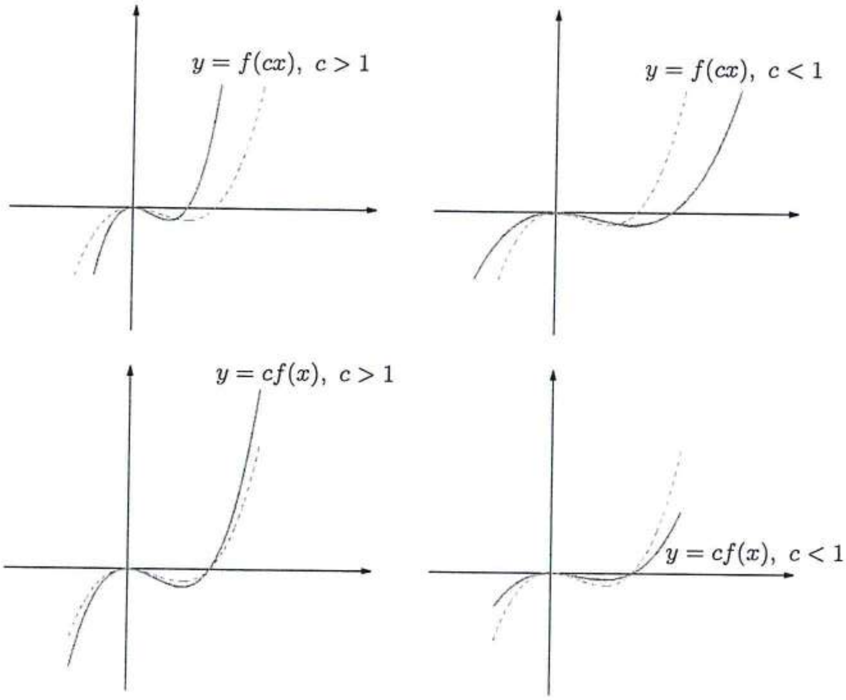

Figure 2.13. Graph of $f(cx)$ with $c > 1$ (top left), $0 < c < 1$ (top right), and of $cf(x)$ with $c > 1$ (bottom left), $0 < c < 1$ (bottom right)

2.6 Elementary functions and properties

We start with a few useful definitions.

Definition 2.11 *Let $f : \operatorname{dom} f \subseteq \mathbb{R} \to \mathbb{R}$ be a map with a symmetric domain with respect to the origin, hence such that $x \in \operatorname{dom} f$ forces $-x \in \operatorname{dom} f$ as well. The function f is said* **even** *if $f(-x) = f(x)$ for all $x \in \operatorname{dom} f$,* **odd** *if $f(-x) = -f(x)$ for all $x \in \operatorname{dom} f$.*

The graph of an even function is symmetric with respect to the y-axis, and that of an odd map symmetric with respect to the origin. If f is odd and defined in the origin, necessarily it must vanish at the origin, for $f(0) = -f(0)$.

Definition 2.12 *A function $f : \operatorname{dom} f \subseteq \mathbb{R} \to \mathbb{R}$ is said* **periodic of period** *p (with $p > 0$ real) if $\operatorname{dom} f$ is invariant under translations by $\pm p$ (i.e., if $x \pm p \in \operatorname{dom} f$ for all $x \in \operatorname{dom} f$) and if $f(x + p) = f(x)$ holds for any $x \in \operatorname{dom} f$.*

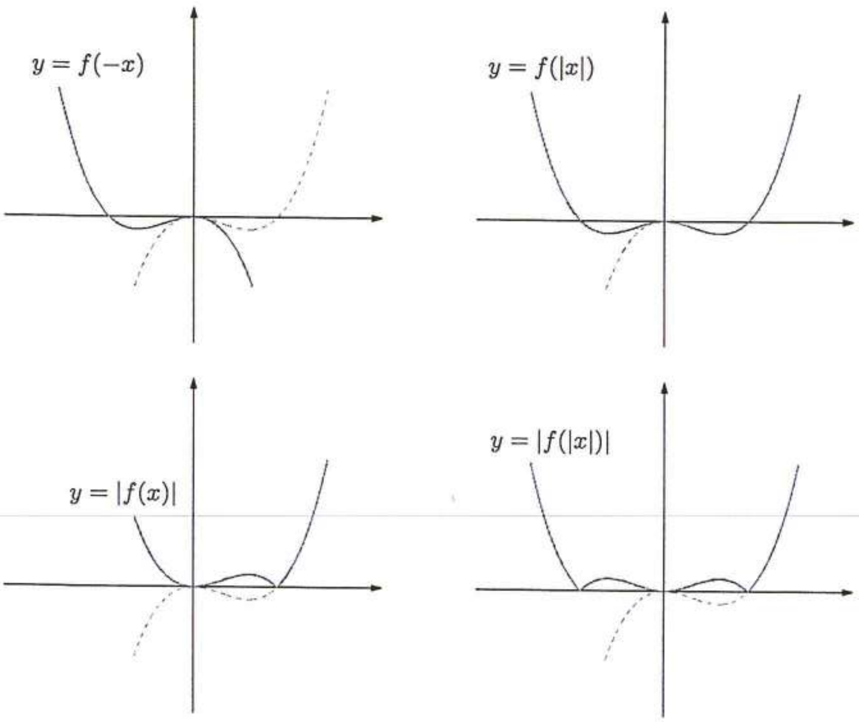

Figure 2.14. Clockwise: graph of the functions $f(-x)$, $f(|x|)$, $|f(|x|)|$, $|f(x)|$

One easily sees that an f periodic of period p is also periodic of any multiple mp ($m \in \mathbb{N} \setminus \{0\}$) of p. If the smallest period exists, it goes under the name of **minimum period** of the function. A constant map is clearly periodic of any period $p > 0$ and thus has no minimum period.

Let us review now the main elementary functions.

2.6.1 Powers

These are functions of the form $y = x^\alpha$. The case $\alpha = 0$ is trivial, giving rise to the constant function $y = x^0 = 1$. Suppose then $\alpha > 0$. For $\alpha = n \in \mathbb{N} \setminus \{0\}$, we find the monomial functions $y = x^n$ defined on $\mathbb{R}$, already considered in Example 2.7 ii) and iii). When n is odd, the maps are odd, strictly increasing on $\mathbb{R}$ and with range $\mathbb{R}$ (recall Property 1.8). When n is even, the functions are even, strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$; their range is the interval $[0, +\infty)$.

Consider now the case $\alpha > 0$ rational. If $\alpha = \frac{1}{m}$ where $m \in \mathbb{N} \setminus \{0\}$, we define a function, called mth root of x and denoted $y = x^{1/m} = \sqrt[m]{x}$, inverting $y = x^m$. It has domain $\mathbb{R}$ if m is odd, $[0, +\infty)$ if m is even. The mth root is strictly increasing and ranges over $\mathbb{R}$ or $[0, +\infty)$, according to whether m is even or odd respectively.

In general, for $\alpha = \frac{n}{m} \in \mathbb{Q}$, $n, m \in \mathbb{N} \setminus \{0\}$ with no common divisors, the function $y = x^{n/m}$ is defined as $y = (x^n)^{1/m} = \sqrt[m]{x^n}$. As such, it has domain $\mathbb{R}$

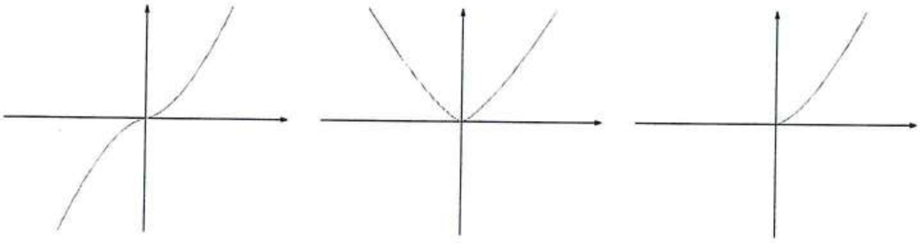

Figure 2.15. Graphs of the functions $y = x^{5/3}$ (left), $y = x^{4/3}$ (middle) and $y = x^{3/2}$ (right)

if m is odd, $[0, +\infty)$ if m is even. It is strictly increasing on $[0, +\infty)$ for any n, m, while if m is odd it strictly increases or decreases on $(-\infty, 0]$ according to the parity of n.

Let us consider some examples (Fig. 2.15). The map $y = x^{5/3}$, defined on $\mathbb{R}$, is strictly increasing and has range $\mathbb{R}$. The map $y = x^{4/3}$ is defined on $\mathbb{R}$, strictly decreases on $(-\infty, 0]$ and strictly increases on $[0, +\infty)$, which is also its range. To conclude, $y = x^{3/2}$ is defined only on $[0, +\infty)$, where it is strictly increasing and has $[0, +\infty)$ as range.

Let us introduce now the generic function $y = x^\alpha$ with irrational $\alpha > 0$. To this end, note that if a is a non-negative real number we can define the power a^α with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$, starting from powers with rational exponent and exploiting the density of rationals inside $\mathbb{R}$. If $a \geq 1$, we can in fact define $a^\alpha = \sup\{a^{n/m} \mid \frac{n}{m} \leq \alpha\}$, while for $0 \leq a < 1$ we set $a^\alpha = \inf\{a^{n/m} \mid \frac{n}{m} \leq \alpha\}$. Thus the map $y = x^\alpha$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ is defined on $[0, +\infty)$, and one proves it is there strictly increasing and its range is $[0, +\infty)$.

Summarising, we have defined $y = x^\alpha$ for every value $\alpha > 0$. They are all defined a least on $[0, +\infty)$, interval on which they are strictly increasing; moreover, they satisfy $y(0) = 0$, $y(1) = 1$. It will turn out useful to remark that if $\alpha < \beta$,

$$0 < x^\beta < x^\alpha < 1, \quad \text{for } 0 < x < 1, \qquad 1 < x^\alpha < x^\beta, \quad \text{for } x > 1 \tag{2.10}$$

(see Fig. 2.16).

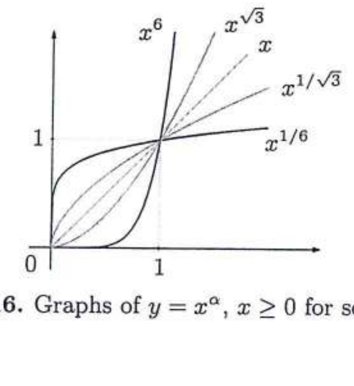

Figure 2.16. Graphs of $y = x^\alpha$, $x \geq 0$ for some $\alpha > 0$

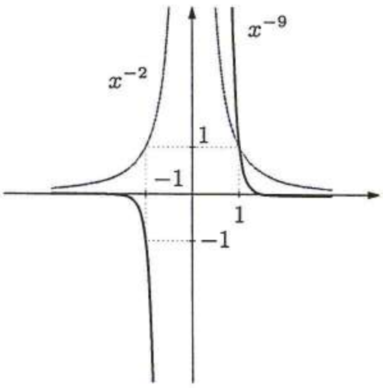

Figure 2.17. Graphs of $y = x^\alpha$ for a two values $\alpha < 0$

At last, consider the case of $\alpha < 0$. Set $y = x^\alpha = \dfrac{1}{x^{-\alpha}}$ by definition. Its domain coincides with the domain of $y = x^{-\alpha}$ minus the origin. All maps are strictly decreasing on $(0, +\infty)$, while on $(-\infty, 0)$ the behaviour is as follows: writing $\alpha = -\frac{n}{m}$ with m odd, the map is strictly increasing if n is even, strictly decreasing if n is odd, as shown in Fig. 2.17. In conclusion, we observe that for every $\alpha \neq 0$, the inverse function of $y = x^\alpha$, where defined, is $y = x^{1/\alpha}$.

2.6.2 Polynomial and rational functions

A **polynomial function**, or simply, a **polynomial**, is a map of the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$; n is the **degree** of the polynomial. Such a map is defined over all $\mathbb{R}$; it is even (resp. odd) if and only if all coefficients indexed by even (odd) subscripts vanish (recall that 0 is an even number).

A **rational function** is of the kind $R(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials. If these have no common factor, the domain of the rational function will be $\mathbb{R}$ without the zeroes of the denominator.

2.6.3 Exponential and logarithmic functions

Let a be a positive real number. According to what we have discussed previously, the **exponential function** $y = a^x$ is defined for any real number x; it satisfies $y(0) = a^0 = 1$.

If $a > 1$, the exponential is strictly increasing; if $a = 1$, this is the constant map 1, while if $a < 1$, the function is strictly decreasing. When $a \neq 1$, the range is $(0, +\infty)$ (Fig. 2.18). Recalling a few properties of powers is useful at this point: for any $x, y \in \mathbb{R}$

$$a^{x+y} = a^x a^y, \qquad a^{x-y} = \frac{a^x}{a^y}, \qquad (a^x)^y = a^{xy}.$$

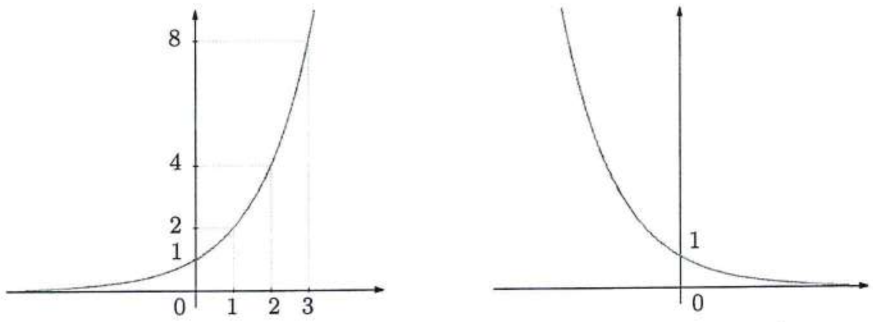

Figure 2.18. Graphs of the exponential functions $y = 2^x$ (left) and $y = (\frac{1}{2})^x$ (right)

When $a \neq 1$, the exponential function is strictly monotone on $\mathbb{R}$, hence invertible. The inverse $y = \log_a x$ is called **logarithm**, is defined on $(0, +\infty)$ and ranges over $\mathbb{R}$; it satisfies $y(1) = \log_a 1 = 0$. The logarithm is strictly increasing if $a > 1$, strictly decreasing if $a < 1$ (Fig. 2.19). The previous properties translate into the following:

$$\log_a(xy) = \log_a x + \log_a y, \quad \forall x, y > 0\,,$$
$$\log_a \frac{x}{y} = \log_a x - \log_a y, \quad \forall x, y > 0\,,$$
$$\log_a(x^y) = y \log_a x, \quad \forall x > 0\,,\ \forall y \in \mathbb{R}\,.$$

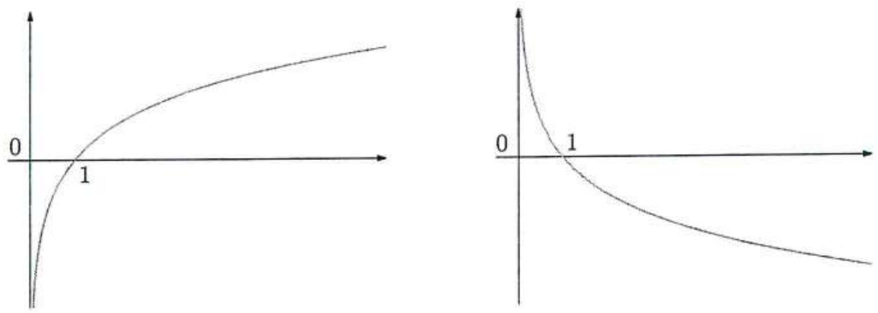

Figure 2.19. Graphs of $y = \log_2 x$ (left) and $y = \log_{1/2} x$ (right)

2.6.4 Trigonometric functions and inverses

Denote here by X, Y the coordinates on the Cartesian plane $\mathbb{R}^2$, and consider the **unit circle**, i.e., the circle of unit radius centred at the origin $O = (0, 0)$, whose equation reads $X^2 + Y^2 = 1$. Starting from the point $A = (1, 0)$, intersection of the circle with the positive x-axis, we go around the circle. More precisely, given any real x we denote by $P(x)$ the point on the circle reached by turning counter-clockwise along an arc of length x if $x \geq 0$, or clockwise by an arc of length $-x$ if $x < 0$. The point $P(x)$ determines an *angle* in the plane with vertex O and delimited by the outbound rays from O through the points A and $P(x)$ respectively (Fig. 2.20). The number x represents the measure of the angle in *radians*. The one-radian angle is determined by an arc of length 1. This angle measures $\frac{360}{2\pi} = 57.2957795\cdots$ degrees. Table 2.1 provides the correspondence between degrees and radians for important angles. Henceforth all angles shall be expressed in radians without further mention.

degrees	0	30	45	60	90	120	135	150	180	270	360
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Table 2.1. Degrees versus radians

Increasing or decreasing by 2π the length x has the effect of going around the circle once, counter-clockwise or clockwise respectively, and returning to the initial point $P(x)$. In other words, there is a periodicity

$$P(x \pm 2\pi) = P(x), \qquad \forall x \in \mathbb{R}. \tag{2.11}$$

Denote by $\cos x$ ('cosine of x') and $\sin x$ ('sine of x') the X- and Y-coordinates, respectively, of the point $P(x)$. Thus $P(x) = (\cos x, \sin x)$. Hence the **cosine function** $y = \cos x$ and the **sine function** $y = \sin x$ are defined on $\mathbb{R}$ and assume all

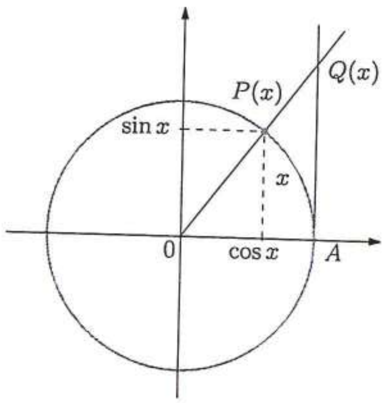

Figure 2.20. The unit circle

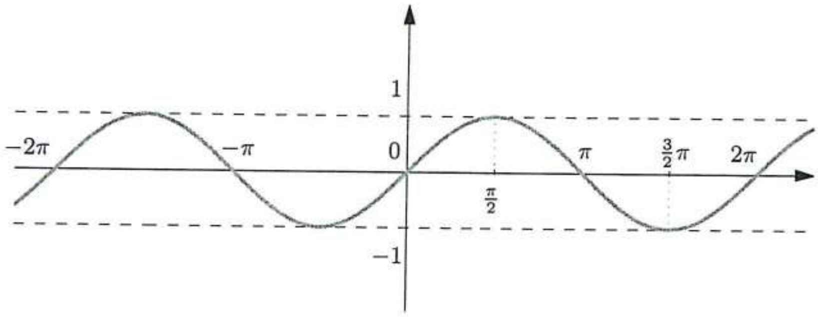

Figure 2.21. Graph of the map $y = \sin x$

values of the interval $[-1, 1]$; by (2.11), they are periodic maps of minimum period 2π. They satisfy the crucial trigonometric relation

$$\boxed{\cos^2 x + \sin^2 x = 1, \qquad \forall x \in \mathbb{R}.} \tag{2.12}$$

It is rather evident from the geometric interpretation that the sine function is odd, while the cosine function is even. Their graphs are represented in Figures 2.21 and 2.22.

Important values of these maps are listed in the following table (where k is any integer):

$\sin x = 0 \quad \text{for } x = k\pi\,,$	$\cos x = 0 \quad \text{for } x = \frac{\pi}{2} + k\pi\,,$
$\sin x = 1 \quad \text{for } x = \frac{\pi}{2} + 2k\pi\,,$	$\cos x = 1 \quad \text{for } x = 2k\pi\,,$
$\sin x = -1 \quad \text{for } x = -\frac{\pi}{2} + 2k\pi\,,$	$\cos x = -1 \quad \text{for } x = \pi + 2k\pi\,.$

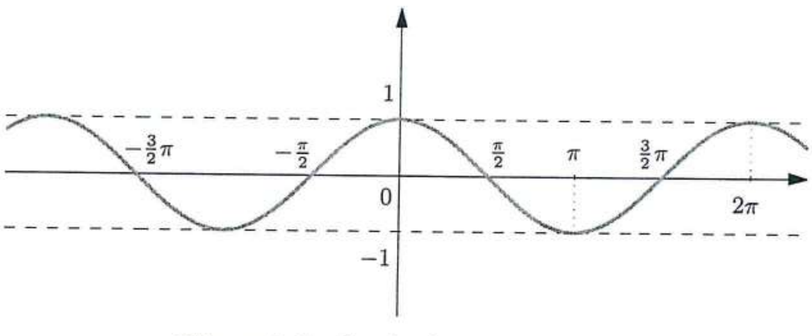

Figure 2.22. Graph of the map $y = \cos x$

Concerning monotonicity, one has

$$y = \sin x \quad \text{is} \quad \begin{cases} \text{strictly increasing on} & \left[-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right] \\ \text{strictly decreasing on} & \left[\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right], \end{cases}$$

$$y = \cos x \quad \text{is} \quad \begin{cases} \text{strictly decreasing on} & [2k\pi, \pi + 2k\pi] \\ \text{strictly increasing on} & [\pi + 2k\pi, 2\pi + 2k\pi]. \end{cases}$$

The addition and subtraction formulas are relevant

$$\boxed{\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \end{aligned}}$$

Suitable choices of the arguments allow to infer from these the duplication formulas

$$\boxed{\sin 2x = 2\sin x\cos x, \qquad \cos 2x = 2\cos^2 x - 1,} \tag{2.13}$$

rather than

$$\boxed{\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2},} \tag{2.14}$$

$$\boxed{\cos x - \cos y = -2\sin\frac{x-y}{2}\sin\frac{x+y}{2},} \tag{2.15}$$

or the following

$$\boxed{\sin(x+\pi) = -\sin x, \qquad \cos(x+\pi) = -\cos x,} \tag{2.16}$$

$$\boxed{\sin(x+\frac{\pi}{2}) = \cos x, \qquad \cos(x+\frac{\pi}{2}) = -\sin x.} \tag{2.17}$$

In the light of Sect. 2.5.1, the first of (2.17) tells that the graph of the cosine is obtained by left-translating the sine's graph by $\pi/2$ (compare Figures 2.21 and 2.22).

The **tangent function** $y = \tan x$ (sometimes $y = \operatorname{tg} x$) and the **cotangent function** $y = \operatorname{cotan} x$ (also $y = \operatorname{ctg} x$) are defined by

$$\boxed{\tan x = \frac{\sin x}{\cos x}, \qquad \operatorname{cotan} x = \frac{\cos x}{\sin x}.}$$

Because of (2.16), these maps are periodic of minimum period π, and not 2π. The tangent function is defined on $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z}\}$, it is strictly increasing on the

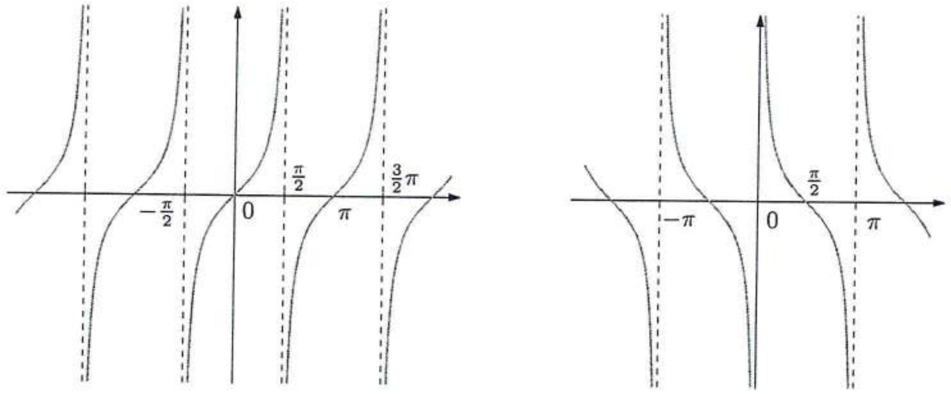

Figure 2.23. Graphs of the functions $y = \tan x$ (left) and $y = \operatorname{cotan} x$ (right)

intervals $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$ where it assumes every real number as value. Similarly, the cotangent function is defined on $\mathbb{R} \setminus \{k\pi \ : \ k \in \mathbb{Z}\}$, is strictly decreasing on the intervals $(k\pi, \pi + k\pi)$, on which it assumes every real value. Both maps are odd. Their respective graphs are found in Fig. 2.23.

Recall that $\tan x$ expresses geometrically the Y-coordinate of the intersection point $Q(x)$ between the ray from the origin through $P(x)$ and the vertical line containing A (Fig. 2.20).

The trigonometric functions, being periodic, cannot be invertible on their whole domains. In order to invert them, one has to restrict to a maximal interval of strict monotonicity; in each case one such interval is chosen.

The map $y = \sin x$ is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The inverse function on this particular interval is called **inverse sine** or **arcsine** and denoted $y = \arcsin x$

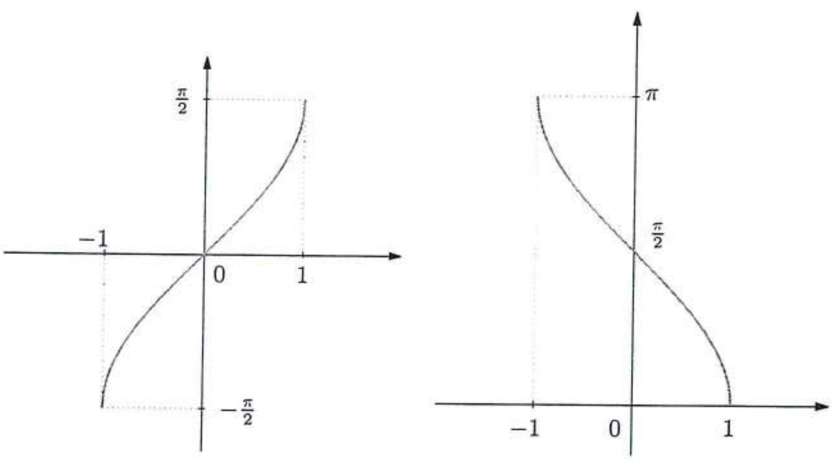

Figure 2.24. Graphs of $y = \arcsin x$ (left) and $y = \arccos x$ (right)

or $y = \operatorname{asin} x$; it is defined on $[-1, 1]$, everywhere strictly increasing and ranging over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This function is odd (Fig. 2.24, left).

Similarly, the function $y = \cos x$ is strictly decreasing on the interval $[0, \pi]$. By restricting it to this interval one can define the **inverse cosine**, or **arccosine**, $y = \arccos x$ or $y = \operatorname{acos} x$ on $[-1, 1]$, which is everywhere strictly decreasing and has $[0, \pi]$ for range (Fig. 2.24, right).

The function $y = \tan x$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. There, the inverse is called **inverse tangent**, or **arctangent**, and denoted $y = \arctan x$ or $y = \operatorname{atan} x$ (also $\operatorname{arctg} x$). It is strictly increasing on its entire domain $\mathbb{R}$, and has range $(-\frac{\pi}{2}, \frac{\pi}{2})$. Also this is an odd map (Fig. 2.25, left).

In the analogous way the **inverse cotangent**, or **arccotangent**, $y = \operatorname{arccotan} x$ is the inverse of the cotangent on $(0, \pi)$ (Fig. 2.25, right).

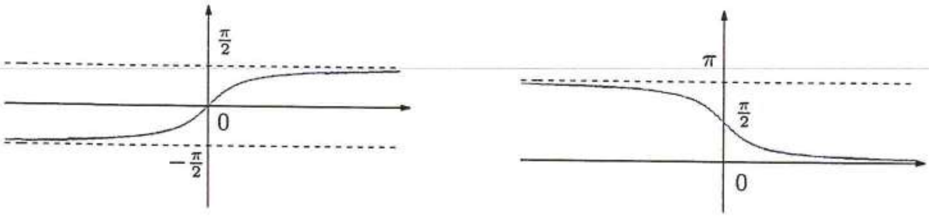

Figure 2.25. Graphs of $y = \arctan x$ (left) and $y = \operatorname{arccotan} x$ (right)

2.7 Exercises

1. *Determine the domains of the following functions:*

a) $f(x) = \dfrac{3x+1}{x^2+x-6}$

[b)] $f(x) = \dfrac{\sqrt{x^2-3x-4}}{x+5}$

c) $f(x) = \log(x^2 - x)$

[d)] $f(x) = \begin{cases} \dfrac{1}{2x+1} & \text{if } x \ge 0\,, \\ e^{\sqrt{x+1}} & \text{if } x < 0 \end{cases}$

2. *Determine the range of the following functions:*

[a)] $f(x) = \dfrac{1}{x^2+1}$

[b)] $f(x) = \sqrt{x+2} - 1$

c) $f(x) = e^{5x+3}$

d) $f(x) = \begin{cases} \log x & \text{if } x \ge 1\,, \\ -2x-5 & \text{if } x < 1 \end{cases}$

[3.] *Find domain and range for the map* $f(x) = \sqrt{\cos x - 1}$ *and plot its graph.*

4. *Let* $f(x) = -\log(x-1)$; *determine* $f^{-1}([0,+\infty))$ *and* $f^{-1}((-\infty,-1])$.

5. *Sketch the graph of the following functions indicating the possible symmetries and/or periodicity:*

   a) $f(x) = \sqrt{1-|x|}$ b) $f(x) = 1 + \cos 2x$

   c) $f(x) = \tan\left(x + \frac{\pi}{2}\right)$ d) $f(x) = \begin{cases} x^2 - x - 1 & \text{if } x \leq 1\,, \\ -x & \text{if } x > 1 \end{cases}$

6. *Using the map* $f(x)$ *in Fig. 2.26, draw the graphs of*

$$f(x) - 1, \quad f(x+3), \quad f(x-1), \quad -f(x), \quad f(-x), \quad |f(x)|.$$

7. *Check that the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2 - 2x + 5$ *is not invertible. Determine suitable invertible restrictions of* f *and write down the inverses explicitly.*

8. *Determine the largest interval* I *where the map*

$$f(x) = \sqrt{|x-2| - |x| + 2}$$

is invertible, and plot a graph. Write the expression of the inverse function of f *restricted to* I.

9. *Verify that* $f(x) = (1+3x)(2x - |x-1|)$, *defined on* $[0,+\infty)$, *is one-to-one. Determine its range and inverse function.*

10. *Let* f *and* g *be the functions below. Write the expressions for* $g \circ f$, $f \circ g$, *and determine the composites' domains.*

    a) $f(x) = x^2 - 3$ and $g(x) = \log(1+x)$

    b) $f(x) = \dfrac{7x}{x+1}$ and $g(x) = \sqrt{2-x}$

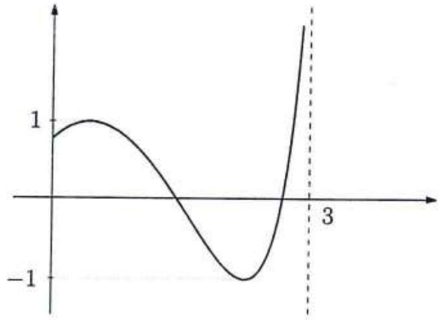

Figure 2.26. Graph of the function f in Exercise 6

11. *Write* $h(x) = \dfrac{2e^x + 1}{e^{2x} + 2}$ *as composition of the map* $f(x) = e^x$ *with some other function.*

12. *Given* $f(x) = x^2 - 3x + 2$ *and* $g(x) = x^2 - 5x + 6$, *find the expressions and graphs of*

$$h(x) = \min(f(x), g(x)) \qquad \text{and} \qquad k(x) = \max(h(x), 0).$$

2.7.1 Solutions

1. *Domains:*

a) $\operatorname{dom} f = \mathbb{R} \setminus \{-3, 2\}$.

b) The conditions $x^2 - 3x - 4 \geq 0$ and $x + 5 \neq 0$ are necessary. The first is tantamount to $(x+1)(x-4) \geq 0$, hence $x \in (-\infty, -1] \cup [4, +\infty)$; the second to $x \neq -5$. The domain of f is then

$$\operatorname{dom} f = (-\infty, -5) \cup (-5, -1] \cup [4, +\infty).$$

c) $\operatorname{dom} f = (-\infty, 0) \cup (1, +\infty)$.

d) In order to study the domain of this piecewise function, we treat the cases $x \geq 0$, $x < 0$ separately.
For $x \geq 0$, we must impose $2x+1 \neq 0$, i.e., $x \neq -\frac{1}{2}$. Since $-\frac{1}{2} < 0$, the function is well defined on $x \geq 0$.
For $x < 0$, we must have $x + 1 \geq 0$, or $x \geq -1$. For negative x then, the function is defined on $[-1, 0)$.
All in all, $\operatorname{dom} f = [-1, +\infty)$.

2. *Ranges:*

a) The map $y = x^2$ has range $[0, +\infty)$; therefore the range of $y = x^2 + 1$ is $[1, +\infty)$. Passing to reciprocals, the given function ranges over $(0, 1]$.

b) The map is obtained by translating the elementary function $y = \sqrt{x}$ (whose range is $[0, +\infty)$) to the left by -2 (yielding $y = \sqrt{x+2}$) and then downwards by 1 (which gives $y = \sqrt{x+2} - 1$). The graph is visualised in Fig. 2.27, and clearly $\operatorname{im} f = [-1, +\infty)$.
Alternatively, one can observe that $0 \leq \sqrt{x+2} < +\infty$ implies $-1 \leq \sqrt{x+2} - 1 < +\infty$, whence $\operatorname{im} f = [-1, +\infty)$.

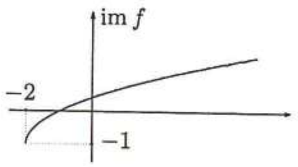

Figure 2.27. Graph of $y = \sqrt{x+2} - 1$

c) $\operatorname{im} f = (0, +\infty)$; d) $\operatorname{im} f = (-7, +\infty)$.

3. Imposing $\cos x - 1 \geq 0$ tells that $\cos x \geq 1$. Such constraint is true only for $x = 2k\pi$, $k \in \mathbb{Z}$, where the cosine equals 1; thus $\operatorname{dom} f = \{x \in \mathbb{R} : x = 2k\pi,\ k \in \mathbb{Z}\}$ and $\operatorname{im} f = \{0\}$. Fig. 2.28 provides the graph.

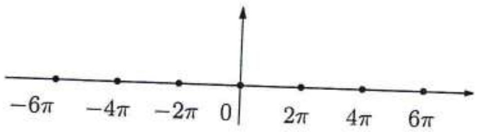

Figure 2.28. Graph of $y = \sqrt{\cos x - 1}$

4. $f^{-1}([0, +\infty)) = (1, 2]$ and $f^{-1}((-\infty, -1]) = [e + 1, +\infty)$.

5. *Graphs and symmetries/periodicity:*

a) The function is even, not periodic and its graph is shown in Fig. 2.29 (top left).
b) The map is even and periodic of period π, with graph in Fig. 2.29 (top right).
c) This function is odd and periodic with period π, see Fig. 2.29 (bottom left).
d) The function has no symmetries nor a periodic behaviour, as shown in Fig. 2.29 (bottom right).

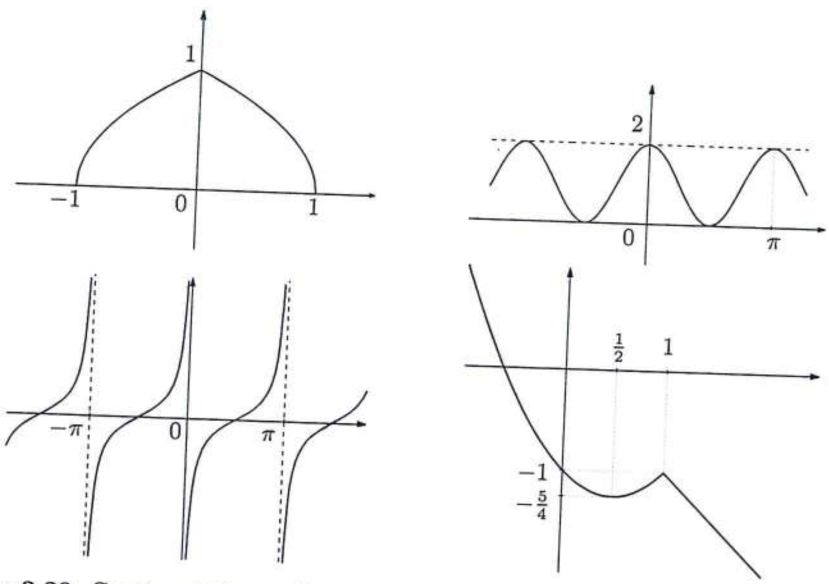

Figure 2.29. Graphs relative to Exercises 5.a) (top left), 5.b) (top right), 5.c) (bottom left) and 5.d) (bottom right)

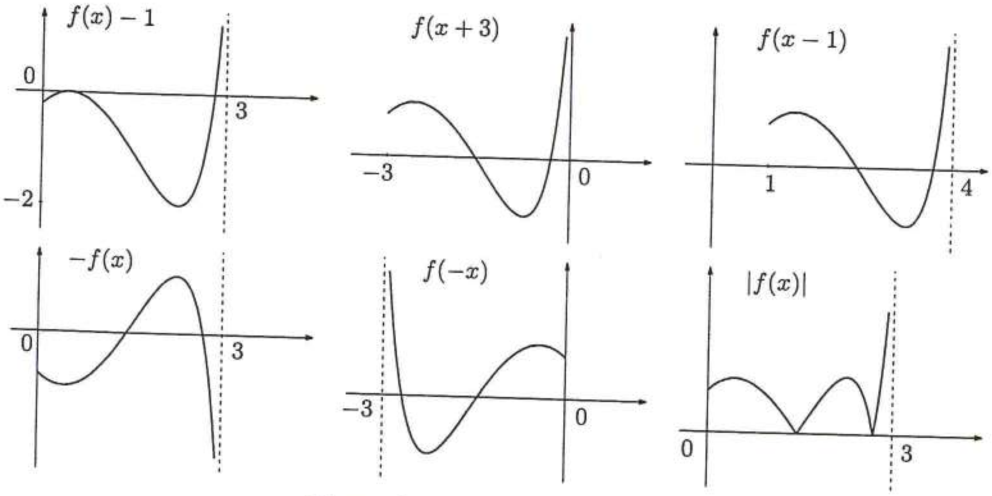

Figure 2.30. Graphs of Exercise 6

6. See Fig. 2.30.

7. The function represents a parabola with vertex $(1,4)$, and as such it is not invertible on $\mathbb{R}$, not being one-to-one (e.g., $f(0) = f(2) = 5$). But restricted to the intervals $(-\infty, 1]$ and $[1, +\infty)$ separately, it becomes invertible. Setting

$$f_1 = f_{|(-\infty,1]} : (-\infty, 1] \to [4, +\infty)\,, \qquad f_2 = f_{|[1,+\infty)} : [1, +\infty) \to [4, +\infty)\,,$$

we can compute

$$f_1^{-1} : [4, +\infty) \to (-\infty, 1]\,, \qquad f_2^{-1} : [4, +\infty) \to [1, +\infty)$$

explicitly. In fact, from $x^2 - 2x + 5 - y = 0$ we obtain

$$x = 1 \pm \sqrt{y-4}.$$

With the ranges of f_1^{-1} and f_2^{-1} in mind, swapping the variables x, y yields

$$f_1^{-1}(x) = 1 - \sqrt{x-4}\,, \qquad f_2^{-1}(x) = 1 + \sqrt{x-4}.$$

8. Since

$$f(x) = \begin{cases} 2 & \text{if } x \le 0\,, \\ \sqrt{4-2x} & \text{if } 0 < x \le 2\,, \\ 0 & \text{if } x > 2\,, \end{cases}$$

the required interval I is $[0, 2]$, and the graph of f is shown in Fig. 2.31.
In addition $f([0, 2]) = [0, 2]$, so $f^{-1} : [0, 2] \to [0, 2]$. By putting $y = \sqrt{4-2x}$ we obtain $x = \frac{4-y^2}{2}$, which implies $f^{-1}(x) = 2 - \frac{1}{2}x^2$.

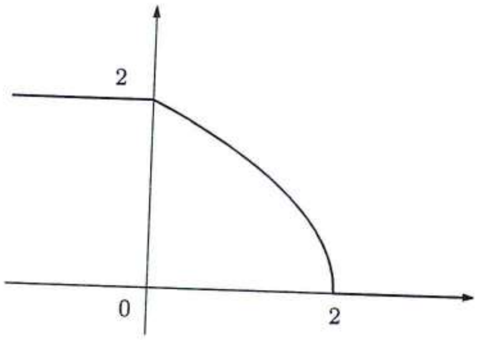

Figure 2.31. Graph of $y = \sqrt{|x-2| - |x| + 2}$

9. We have

$$f(x) = \begin{cases} 9x^2 - 1 & \text{if } 0 \le x \le 1\,, \\ 3x^2 + 4x + 1 & \text{if } x > 1 \end{cases}$$

and the graph of f is in Fig. 2.32.
The range of f is $[-1, +\infty)$. To determine f^{-1} we discuss the cases $0 \le x \le 1$ and $x > 1$ separately. For $0 \le x \le 1$, we have $-1 \le y \le 8$ and

$$y = 9x^2 - 1 \qquad \Longleftrightarrow \qquad x = \sqrt{\frac{y+1}{9}}.$$

For $x > 1$, we have $y > 8$ and

$$y = 3x^2 + 4x + 1 \qquad \Longleftrightarrow \qquad x = \frac{-2 + \sqrt{3y+1}}{3}.$$

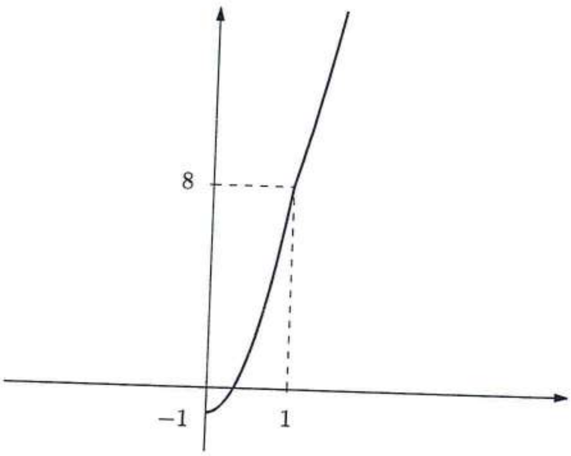

Figure 2.32. Graph of $y = (1+3x)(2x - |x-1|)$

Thus

$$f^{-1}(x) = \begin{cases} \sqrt{\dfrac{x+1}{9}} & \text{if } -1 \le x \le 8\,, \\ \dfrac{-2+\sqrt{3x+1}}{3} & \text{if } x > 8. \end{cases}$$

10. *Composite functions:*

a) As $g \circ f(x) = g(f(x)) = g(x^2-3) = \log(1+x^2-3) = \log(x^2-2)$, it follows $\operatorname{dom} g \circ f = \{x \in \mathbb{R} : x^2 - 2 > 0\} = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$.

We have $f \circ g(x) = f(g(x)) = f(\log(1+x)) = (\log(1+x))^2 - 3$, so $\operatorname{dom} f \circ g = \{x \in \mathbb{R} : 1 + x > 0\} = (-1, +\infty)$.

b) $g \circ f(x) = \sqrt{\dfrac{2-5x}{x+1}}$ and $\operatorname{dom} g \circ f = (-1, \frac{2}{5}]$;

$f \circ g(x) = \dfrac{7\sqrt{2-x}}{\sqrt{2-x}+1}$ and $\operatorname{dom} f \circ g = (-\infty, 2]$.

11. $g(x) = \dfrac{2x+1}{x^2+2}$ and $h(x) = g \circ f(x)$.

12. After drawing the parabolic graphs $f(x)$ and $g(x)$ (Fig. 2.33), one sees that

$$h(x) = \begin{cases} x^2 - 3x + 2 & \text{if } x \le 2\,, \\ x^2 - 5x + 6 & \text{if } x > 2\,, \end{cases}$$

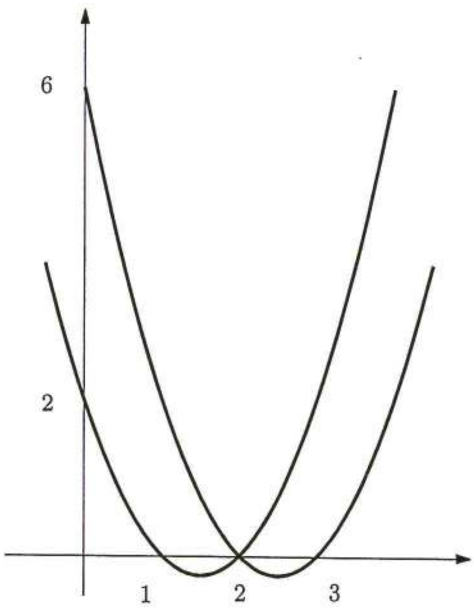

Figure 2.33. Graphs of the parabolas $f(x) = x^2 - 3x + 2$ and $g(x) = x^2 - 5x + 6$