3

Limits and continuity I

This chapter tackles the limit behaviour of a real sequence or a function of one real variable, and studies the continuity of such a function.

3.1 Neighbourhoods

The process of defining limits and continuity leads to consider real numbers which are 'close' to a certain real number. In equivalent geometrical jargon, one considers points on the real line 'in the proximity' of a given point. Let us begin by making mathematical sense of the notion of neighbourhood of a point.

Definition 3.1 *Let $x_0 \in \mathbb{R}$ be a point on the real line, and $r > 0$ a real number. We call* **neighbourhood of x_0 of radius** r *the open and bounded interval*

$$I_r(x_0) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R} \ : \ |x - x_0| < r\}.$$

Hence, the neighbourhood of 2 of radius 10^{-1}, denoted $I_{10^{-1}}(2)$, is the set of real numbers lying between 1.9 and 2.1, these excluded. By understanding the quantity $|x - x_0|$ as the Euclidean **distance** between the points x_0 and x, we can then say that $I_r(x_0)$ consists of the points on the real line whose distance from x_0 is less than r. If we interpret $|x - x_0|$ as the **tolerance** in the approximation of x_0 by x, then $I_r(x_0)$ becomes the set of real numbers approximating x_0 with a better margin of precision than r.

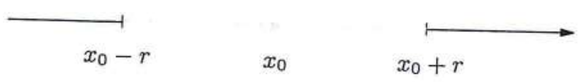


Figure 3.1. Neighbourhood of x_0 of radius r

C. Canuto, A. Tabacco: *Mathematical Analysis I*, 2nd Ed.,
UNITEXT – La Matematica per il 3+2 84, DOI 10.1007/978-3-319-12772-9_3,

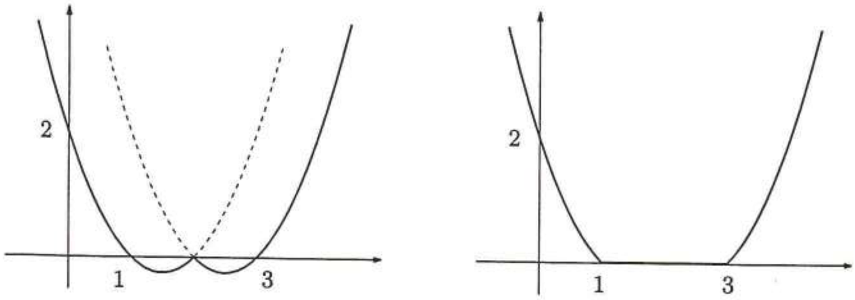


Figure 2.34. Graphs of the maps h (left) and k (right) relative to Exercise 12

and the graph of h is that of Fig. 2.34, left.
Proceeding as above,

$$k(x) = \begin{cases} x^2 - 3x + 2 & \text{if } x \le 1\,, \\ 0 & \text{if } 1 < x < 3\,, \\ x^2 - 5x + 6 & \text{if } x \ge 3\,, \end{cases}$$

and k has a graph as in Fig. 2.34, right.

Varying r in the set of positive real numbers, while mantaining x_0 in $\mathbb{R}$ fixed, we obtain a **family of neighbourhoods** of x_0. Each neighbourhood is a proper subset of any other in the family that has bigger radius, and in turn it contains all neighbourhoods of lesser radius.

Remark 3.2 The notion of neighbourhood of a point $x_0 \in \mathbb{R}$ is nothing but a particular case of the analogue for a point in the Cartesian product $\mathbb{R}^d$ (hence the plane if $d = 2$, space if $d = 3$), presented in Definition 8.11.

The upcoming definitions of limit and continuity, based on the idea of neighbourhood, can be stated directly for functions on $\mathbb{R}^d$, by considering functions of one real variable as subcases for $d = 1$. We prefer to follow a more gradual approach, so we shall examine first the one-dimensional case. Sect. 8.5 will be devoted to explaining how all this generalises to several dimensions. □

It is also convenient to include the case where x_0 is one of the points $+\infty$ or $-\infty$.

Definition 3.3 *For any real* $a \geq 0$*, we call* **neighbourhood of** $+\infty$ **with end-point** a *the open, unbounded interval*

$$I_a(+\infty) = (a, +\infty).$$

Similarly, a **neighbourhood of** $-\infty$ **with end-point** $-a$ *will be defined as*

$$I_a(-\infty) = (-\infty, -a).$$

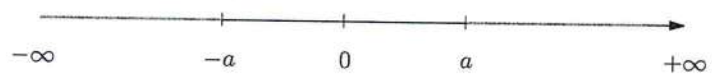


Figure 3.2. Neighbourhoods of $-\infty$ (left) and $+\infty$ (right)

The following convention will be useful in the sequel. We shall say that the property $P(x)$ holds 'in a neighbourhood' of a point c (c being a real number x_0 or $+\infty$, $-\infty$) if there is a certain neighbourhood of c such that for each of its points x, $P(x)$ holds. Colloquially, one also says '$P(x)$ holds around c', especially when the neighbourhood needs not to be specified. For example, the map $f(x) = 2x - 1$ is positive in a neighbourhood of $x_0 = 1$; in fact, $f(x) > 0$ for any $x \in I_{1/2}(1)$.

3.2 Limit of a sequence

Consider a real sequence $a : n \mapsto a_n$. We are interested in studying the behaviour of the values a_n as n increases, and we do so by looking first at a couple of examples.

Examples 3.4

i) Let $a_n = \dfrac{n}{n+1}$. The first terms of this sequence are presented in Table 3.1. We see that the values approach 1 as n increases. More precisely, the real number 1 can be approximated as well as we like by a_n for n sufficiently large. This clause is to be understood in the following sense: however small we fix $\varepsilon > 0$, from a certain point n_ε onwards all values a_n approximate 1 with a margin smaller that ε.

The condition $|a_n - 1| < \varepsilon$, in fact, is tantamount to $\dfrac{1}{n+1} < \varepsilon$, i.e., $n+1 > \dfrac{1}{\varepsilon}$; thus defining $n_\varepsilon = \left[\dfrac{1}{\varepsilon}\right]$ and taking any natural number $n > n_\varepsilon$, we have $n+1 > \left[\dfrac{1}{\varepsilon}\right] + 1 > \dfrac{1}{\varepsilon}$, hence $|a_n - 1| < \varepsilon$. In other words, for every $\varepsilon > 0$, there exists an n_ε such that

$$n > n_\varepsilon \quad \Rightarrow \quad |a_n - 1| < \varepsilon.$$

Looking at the graph of the sequence (Fig. 3.3), one can say that for all $n > n_\varepsilon$ the points (n, a_n) of the graph lie between the horizontal lines $y = 1 - \varepsilon$ and $y = 1 + \varepsilon$.

n	a_n
0	0.00000000000000
1	0.50000000000000
2	0.66666666666667
3	0.75000000000000
4	0.80000000000000
5	0.83333333333333
6	0.85714285714286
7	0.87500000000000
8	0.88888888888889
9	0.90000000000000
10	0.90909090909090
100	0.99009900990099
1000	0.99900099900100
10000	0.99990000999900
100000	0.99999000010000
1000000	0.99999900000100
10000000	0.99999990000001
100000000	0.99999999000000

n	a_n
1	2.0000000000000
2	2.2500000000000
3	2.3703703703704
4	2.4414062500000
5	2.4883200000000
6	2.5216263717421
7	2.5464996970407
8	2.5657845139503
9	2.5811747917132
10	2.5937424601000
100	2.7048138294215
1000	2.7169239322355
10000	2.7181459268244
100000	2.7182682371975
1000000	2.7182804691564
10000000	2.7182816939804
100000000	2.7182817863958

Table 3.1. Values, estimated to the 14th digit, of the sequences $a_n = \frac{n}{n+1}$ (left) and $a_n = \left(1 + \frac{1}{n}\right)^n$ (right)

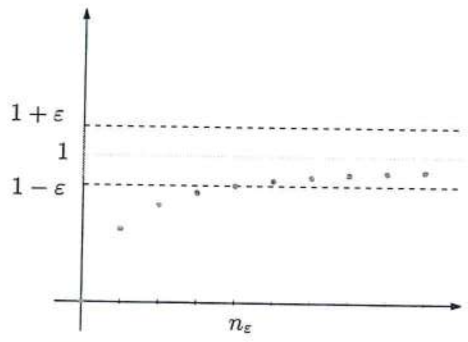


Figure 3.3. Convergence of the sequence $a_n = \frac{n}{n+1}$

ii) The first values of the sequence $a_n = \left(1+\dfrac{1}{n}\right)^n$ are shown in Table 3.1. One could imagine, even expect, that as n increases the values a_n get closer to a certain real number, whose decimal expansion starts as $2.718\ldots$ This is actually the case, and we shall return to this important example later. □

We introduce the notion of converging sequence. For simplicity we shall assume the sequence is defined on the set $\{n \in \mathbb{N} : n \geq n_0\}$ for a suitable $n_0 \geq 0$.

Definition 3.5 *A sequence* $a : n \mapsto a_n$ **converges to the limit** $\ell \in \mathbb{R}$ *(or* **converges to** ℓ *or* **has limit** ℓ*), in symbols*

$$\lim_{n\to\infty} a_n = \ell,$$

if for any real number $\varepsilon > 0$ *there exists an integer* n_ε *such that*

$$\forall n \geq n_0, \qquad n > n_\varepsilon \quad \Rightarrow \quad |a_n - \ell| < \varepsilon.$$

Using the language of neighbourhoods, the condition $n > n_\varepsilon$ can be written $n \in I_{n_\varepsilon}(+\infty)$, while $|a_n - \ell| < \varepsilon$ becomes $a_n \in I_\varepsilon(\ell)$. Therefore, the definition of convergence to a limit is equivalent to: for any neighbourhood $I_\varepsilon(\ell)$ of ℓ, there exists a neighbourhood $I_{n_\varepsilon}(+\infty)$ of $+\infty$ such that

$$\forall n \geq n_0, \qquad n \in I_{n_\varepsilon}(+\infty) \quad \Rightarrow \quad a_n \in I_\varepsilon(\ell).$$

Examples 3.6

i) Referring to Example 3.4 i), we can say

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

ii) Let us check that

$$\lim_{n\to\infty} \frac{3n}{2+5n^2} = 0.$$

Given $\varepsilon > 0$, we must show

$$\left|\frac{3n}{2+5n^2}\right| < \varepsilon$$

for all n greater than a suitable natural number n_ε. Observing that for $n \geq 1$

$$\left|\frac{3n}{2+5n^2}\right| = \frac{3n}{2+5n^2} < \frac{3n}{5n^2} = \frac{3}{5n},$$

we have

$$\frac{3}{5n} < \varepsilon \quad \Rightarrow \quad \left|\frac{3n}{2+5n^2}\right| < \varepsilon.$$

But

$$\frac{3}{5n} < \varepsilon \quad \Longleftrightarrow \quad n > \frac{3}{5\varepsilon},$$

so we can set $n_\varepsilon = \left[\frac{3}{5\varepsilon}\right]$. □

Let us examine now a different behaviour as n increases. Consider for instance the sequence

$$a : n \mapsto a_n = n^2.$$

Its first few values are written in Table 3.2. Not only the values seem not to converge to any finite limit ℓ, they are not even bounded from above: however large we choose a real number $A > 0$, if n is big enough (meaning larger than a suitable n_A), a_n will be bigger than A. In fact, it is sufficient to choose $n_A = [\sqrt{A}]$ and note

$$n > n_A \Rightarrow n > \sqrt{A} \Rightarrow n^2 > A.$$

One says that the sequence diverges to $+\infty$ when that happens.

n	a_n
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
100	10000
1000	1000000
10000	100000000
100000	10000000000

Table 3.2. Values of $a_n = n^2$

In general the notion of divergent sequence is defined as follows.

Definition 3.7 *The sequence* $a : n \mapsto a_n$ **tends to** $+\infty$ (*or* **diverges to** $+\infty$, *or* **has limit** $+\infty$), *written*

$$\lim_{n\to\infty} a_n = +\infty,$$

if for any real $A > 0$ *there exists an* n_A *such that*

$$\forall n \geq n_0, \qquad n > n_A \quad \Rightarrow \quad a_n > A. \tag{3.1}$$

Using neighbourhoods, one can also say that for any neighbourhood $I_A(+\infty)$ of $+\infty$, there is a neighbourhood $I_{n_A}(+\infty)$ of $+\infty$ satisfying

$$\forall n \geq n_0, \qquad n \in I_{n_A}(+\infty) \quad \Rightarrow \quad a_n \in I_A(+\infty).$$

The definition of

$$\lim_{n\to\infty} a_n = -\infty$$

is completely analogous, with the proviso that the implication of (3.1) is changed to

$$\forall n \geq n_0, \qquad n > n_A \quad \Rightarrow \quad a_n < -A.$$

Examples 3.8

i) From what we have seen it is clear that

$$\lim_{n\to\infty} n^2 = +\infty.$$

ii) The sequence $a_n = 0 + 1 + 2 + \ldots + n = \sum_{k=0}^{n} k$ associates to n the sum of the natural numbers up to n. To determine the limit we show first of all that

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}, \tag{3.2}$$

a relation with several uses in Mathematics. For that, note that a_n can also be written as $a_n = n + (n-1) + \ldots + 2 + 1 + 0 = \sum_{k=0}^{n}(n-k)$, hence

$$2a_n = \sum_{k=0}^{n} k + \sum_{k=0}^{n}(n-k) = \sum_{k=0}^{n} n = n\sum_{k=0}^{n} 1 = n(n+1),$$

and the claim follows. Let us verify $\lim_{n\to\infty} \frac{n(n+1)}{2} = +\infty$. Since $\frac{n(n+1)}{2} > \frac{n^2}{2}$, we can proceed as in the example above, so for a given $A > 0$, we may choose $n_A = [\sqrt{2A}]$ □

The previous examples show that some sequences are **convergent**, other **divergent** (to $+\infty$ or $-\infty$). But if neither of these cases occurs, one says that the sequence is **indeterminate**. Such are for instance the sequence $a_n = (-1)^n$, which we have already met, or

$$a_n = \big(1 + (-1)^n\big)\, n = \begin{cases} 2n & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

A sufficient condition to avoid an indeterminate behaviour is **monotonicity**. The definitions concerning monotone functions, given in Sect. 2.4, apply to sequences, as well, which are nothing but particular functions defined over the natural numbers. For them they become particularly simple: it will be enough to compare the values for *all* pairs of subscripts n, $n+1$ belonging to the domain of the sequence. So, a sequence is **monotone increasing** if

$$\boxed{\forall n \geq n_0, \qquad a_n \leq a_{n+1},}$$

the other definitions being analogous. The following result holds.

Theorem 3.9 *A monotone sequence $a : n \mapsto a_n$ is either convergent or divergent. Precisely, in case a_n is increasing:*

i) if the sequence is bounded from above, i.e., there is an upper bound $b \in \mathbb{R}$ such that $a_n \leq b$ for all $n \geq n_0$, then the sequence **converges** *to the supremum ℓ of its image:*

$$\lim_{n\to\infty} a_n = \ell = \sup\{a_n \ : \ n \geq n_0\};$$

ii) if the sequence is not bounded from above, then it diverges to $+\infty$.

In case the sequence is decreasing, the assertions modify in the obvious way.

Proof. Assume first that $\{a_n\}$ is bounded from above, which is to say that $\ell = \sup\{a_n : n \geq n_0\} \in \mathbb{R}$. Due to conditions (1.7), for any $\varepsilon > 0$ there exists an element a_{n_ε} such that $\ell - \varepsilon < a_{n_\varepsilon} \leq \ell$. As the sequence is monotone, $a_{n_\varepsilon} \leq a_n$, $\forall n \geq n_\varepsilon$; moreover, $a_n \leq \ell$, $\forall n \geq n_0$ by definition of the supremum. Therefore

$$\ell - \varepsilon < a_n \leq \ell < \ell + \varepsilon, \qquad \forall n \geq n_\varepsilon,$$

hence each term a_n with $n \geq n_\varepsilon$ belongs to the neighbourhood of ℓ of radius ε. But this is precisely the meaning of

$$\lim_{n\to\infty} a_n = \ell.$$

Let now $\ell = +\infty$. Put differently, for any $A > 0$ there exists an element a_{n_A} so that $a_{n_A} > A$. Monotonicity implies $a_n \geq a_{n_A} > A$, $\forall n \geq n_A$. Thus every a_n with $n \geq n_A$ belongs to the neighbourhood $I_A(+\infty) = (A, +\infty)$ of $+\infty$, i.e.,

$$\lim_{n\to\infty} a_n = +\infty .$$

□

Example 3.10

Let us go back to Example 3.4 i). The sequence $a_n = \dfrac{n}{n+1}$ is strictly increasing, for $a_n < a_{n+1}$, i.e., $\dfrac{n}{n+1} < \dfrac{n+1}{n+2}$, is equivalent to $n(n+2) < (n+1)^2$, hence $n^2 + 2n < n^2 + 2n + 1$, which is valid for any n.

Moreover, $a_n < 1$ for all $n \geq 0$; actually, 1 is the supremum of the set $\{a_n : n \in \mathbb{N}\}$, as remarked in Sect. 1.3.1. Theorem 3.9 recovers the already known result $\lim\limits_{n\to\infty} a_n = 1$. □

The number e

Consider the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$ introduced in Example 3.4 ii). It is possible to prove that it is a strictly increasing sequence (hence in particular $a_n > 2 = a_1$ for any $n > 1$) and that it is bounded from above ($a_n < 3$ for all n). Thus Theorem 3.9 ensures that the sequence converges to a limit between 2 and 3, which traditionally is indicated by the symbol e:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}. \tag{3.3}$$

This number, sometimes called **Napier's number** or **Euler's number**, plays a role of the foremost importance in Mathematics. It is an irrational number, whose first decimal digits are

$$\mathrm{e} = 2.71828182845905\cdots$$

Proofs of the stated properties are given in Appendix A.2.3, p. 437.

The number e is one of the most popular bases for exponentials and logarithms. The exponential function $y = \mathrm{e}^x$ shall sometimes be denoted by $y = \exp x$. The logarithm in base e is called **natural logarithm** and denoted by log or ln, instead of $\log_{\mathrm{e}}$ (for the base-10 logarithm, or decimal logarithm, one uses the capitalised symbol Log).

3.3 Limits of functions; continuity

Let f be a real function of real variable. We wish to describe the behaviour of the dependent variable $y = f(x)$ when the independent variable x 'approaches' a certain point $x_0 \in \mathbb{R}$, or one of the points at infinity $-\infty, +\infty$. We start with the latter case for conveniency, because we have already studied what sequences do at infinity.

3.3.1 Limits at infinity

Suppose f is defined around $+\infty$. In analogy to sequences we have some definitions.

Definition 3.11 *The function f* **tends to the limit** $\ell \in \mathbb{R}$ **for** x **going to** $+\infty$, *in symbols*

$$\lim_{x\to+\infty} f(x) = \ell,$$

if for any real number $\varepsilon > 0$ there is a real $B \geq 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad x > B \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon. \tag{3.4}$$

This condition requires that for any neighbourhood $I_\varepsilon(\ell)$ of ℓ, there exists a neighbourhood $I_B(+\infty)$ of $+\infty$ such that

$$\boxed{\forall x \in \operatorname{dom} f, \qquad x \in I_B(+\infty) \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).}$$

Definition 3.12 *The function f* **tends to** $+\infty$ **for** x **going to** $+\infty$, *in symbols*

$$\lim_{x\to+\infty} f(x) = +\infty,$$

if for each real $A > 0$ there is a real $B \geq 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad x > B \quad \Rightarrow \quad f(x) > A. \tag{3.5}$$

For functions tending to $-\infty$ one should replace $f(x) > A$ by $f(x) < -A$. The expression

$$\boxed{\lim_{x\to+\infty} f(x) = \infty}$$

means $\lim_{x\to+\infty} |f(x)| = +\infty$.

If f is defined around $-\infty$, Definitions 3.11 and 3.12 modify to become definitions of limit (L, finite or infinite) for x going to $-\infty$, by changing $x > B$ to $x < -B$:

$$\boxed{\lim_{x\to-\infty} f(x) = L.}$$

At last, by

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

one intends that f has limit L (finite or not) both for $x \to +\infty$ and $x \to -\infty$.

Examples 3.13

i) Let us check that

$$\lim_{x\to+\infty} \frac{x^2+2x}{2x^2+1} = \frac{1}{2}.$$

Given $\varepsilon > 0$, the condition $|f(x) - \frac{1}{2}| < \varepsilon$ is equivalent to

$$\left| \frac{4x-1}{2(2x^2+1)} \right| < \varepsilon.$$

Without loss of generality we assume $x > \frac{1}{4}$, so that the absolute value sign can be removed. Using simple properties of fractions

$$\frac{4x-1}{2(2x^2+1)} < \frac{2x}{2x^2+1} < \frac{2x}{2x^2} = \frac{1}{x} < \varepsilon \qquad \text{if } x > \frac{1}{\varepsilon}.$$

Thus (3.4) holds for $B = \max\left(\frac{1}{4}, \frac{1}{\varepsilon}\right)$.

ii) We prove

$$\lim_{x\to+\infty} \sqrt{x} = +\infty.$$

Let $A > 0$ be fixed. Since $\sqrt{x} > A$ implies $x > A^2$, putting $B = A^2$ fulfills (3.5).

iii) Consider

$$\lim_{x\to-\infty} \frac{1}{\sqrt{1-x}} = 0.$$

With $\varepsilon > 0$ fixed,

$$\left| \frac{1}{\sqrt{1-x}} \right| = \frac{1}{\sqrt{1-x}} < \varepsilon$$

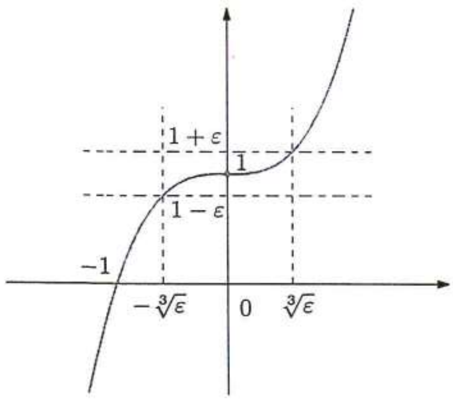
is tantamount to $\sqrt{1-x} > \frac{1}{\varepsilon}$, that is $1 - x > \frac{1}{\varepsilon^2}$, or $x < 1 - \frac{1}{\varepsilon^2}$. So taking $B = \max\left(0, \frac{1}{\varepsilon^2} - 1\right)$, we have

$$x < -B \quad \Rightarrow \quad \left| \frac{1}{\sqrt{1-x}} \right| < \varepsilon. \qquad \square$$

3.3.2 Continuity. Limits at real points

We now investigate the behaviour of the values $y = f(x)$ of a function f when x 'approaches' a point $x_0 \in \mathbb{R}$. Suppose f is defined in a neighbourhood of x_0, but not necessarily at the point x_0 itself. Two examples will let us capture the essence of the notions of continuity and finite limit. Fix $x_0 = 0$ and consider the real functions of real variable $f(x) = x^3 + 1$, $g(x) = x + [1 - x^2]$ and $h(x) = \dfrac{\sin x}{x}$ (recall that $[z]$ indicates the integer part of z); their respective graphs, at least in a neighbourhood of the origin, are presented in Fig. 3.4 and 3.5.

As far as g is concerned, we observe that $|x| < 1$ implies $0 < 1 - x^2 \leq 1$ and g assumes the value 1 only at $x = 0$; in the neighbourhood of the origin of unit radius then,



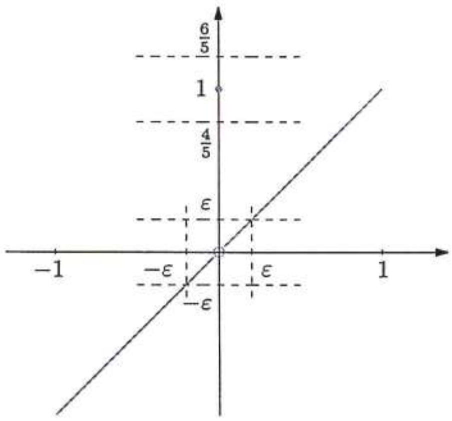


Figure 3.4. Graphs of $f(x) = x^3 + 1$ (left) and $g(x) = x + [1 - x^2]$ (right), in a neighbourhood of the origin

$$g(x) = \begin{cases} 1 & \text{if } x = 0\,, \\ x & \text{if } x \neq 0\,, \end{cases}$$

as the picture shows. Note the function h is not defined in the origin.

For each of f and g, let us compare the values at points x near the origin with the actual value at the origin. The two functions behave rather differently. The value $f(0) = 1$ can be approximated as well as we like by any $f(x)$, provided x is close enough to 0. Precisely, having fixed an (arbitrarily small) 'error' $\varepsilon > 0$ in advance, we can make $|f(x) - f(0)|$ smaller than ε for *all* x such that $|x - 0| = |x|$ is smaller than a suitable real $\delta > 0$. In fact $|f(x) - f(0)| = |x^3| = |x|^3 < \varepsilon$ means $|x| < \sqrt[3]{\varepsilon}$, so it is sufficient to choose $\delta = \sqrt[3]{\varepsilon}$. We shall say that the function f is continuous at the origin.

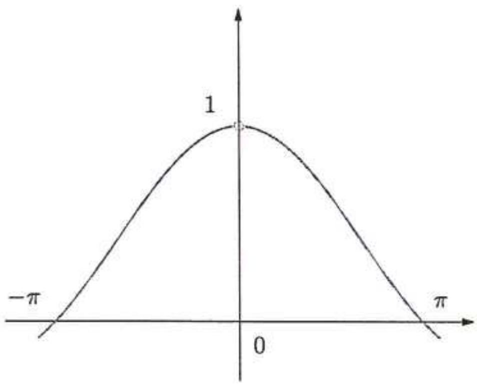


Figure 3.5. Graph of $h(x) = \dfrac{\sin x}{x}$ around the origin

On the other hand, $g(0) = 1$ cannot be approximated well by any $g(x)$ with x close to 0. For instance, let $\varepsilon = \frac{1}{5}$. Then $|g(x) - g(0)| < \varepsilon$ is equivalent to $\frac{4}{5} < g(x) < \frac{6}{5}$; but all x *different from* 0 and such that, say, $|x| < \frac{1}{2}$, satisfy $-\frac{1}{2} < g(x) = x < \frac{1}{2}$, in violation to the constraint for $g(x)$. The function g is not continuous at the origin.

At any rate, we can specify the behaviour of g around 0: for x closer and closer to 0, *yet different from* 0, the images $g(x)$ approximate not the value $g(0)$, but rather $\ell = 0$. In fact, with $\varepsilon > 0$ fixed, if $x \neq 0$ satisfies $|x| < \min(\varepsilon, 1)$, then $g(x) = x$ and $|g(x) - \ell| = |g(x)| = |x| < \varepsilon$. We say that g has limit 0 for x going to 0.

As for the function h, it cannot be continuous at the origin, since comparing the values $h(x)$, for x near 0, with the value at the origin simply makes no sense, for the latter is not even defined. Neverthless, the graph allows to 'conjecture' that these values might estimate $\ell = 1$ increasingly better, the closer we choose x to the origin. We are lead to say h has a limit for x going to 0, and this limit is 1. We shall substantiate this claim later on.

The examples just seen introduce us to the definition of continuity and of (finite) limit.

Definition 3.14 *Let x_0 be a point in the domain of a function f. This function is called* **continuous at** x_0 *if for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\forall x \in \operatorname{dom} f, \qquad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon. \tag{3.6}$$

In neighbourhood-talk: for any neighbourhood $I_\varepsilon(f(x_0))$ of $f(x_0)$ there exists a neighbourhood $I_\delta(x_0)$ of x_0 such that

$$\boxed{\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \quad \Rightarrow \quad f(x) \in I_\varepsilon(f(x_0)).} \tag{3.7}$$

Definition 3.15 *Let f be a function defined on a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. Then f* **has limit** $\ell \in \mathbb{R}$ *(or* **tends to** ℓ *or* **converges to** ℓ*)* **for** x **approaching** x_0*, written*

$$\lim_{x \to x_0} f(x) = \ell,$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon. \tag{3.8}$$

Alternatively: for any given neighbourhood $I_\varepsilon(\ell)$ of ℓ there is a neighbourhood $I_\delta(x_0)$ of x_0 such that

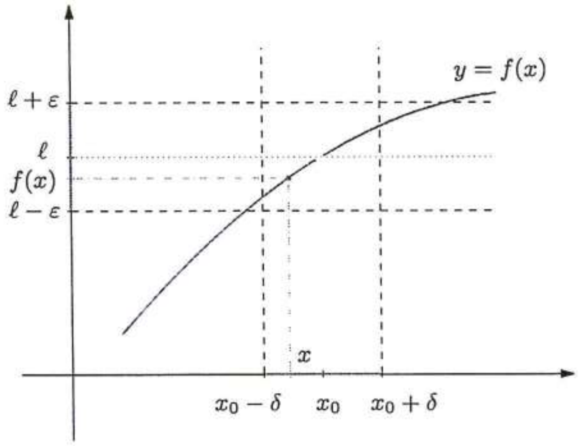


Figure 3.6. Definition of finite limit of a function

$$\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \setminus \{x_0\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

The definition of limit is represented in Fig. 3.6.

Let us compare the notions just seen. To have continuity one looks at the values $f(x)$ from the point of view of $f(x_0)$, whereas for limits these $f(x)$ are compared to ℓ, which *could* be different from $f(x_0)$, provided f is defined in x_0. To test the limit, moreover, the comparison with $x = x_0$ is excluded: requiring $0 < |x - x_0|$ means exactly $x \neq x_0$; on the contrary, the implication (3.6) is obviously true for $x = x_0$.

Let f be defined in a neighbourhood of x_0. If f is continuous at x_0, then (3.8) is certainly true with $\ell = f(x_0)$; vice versa if f has limit $\ell = f(x_0)$ for x going to x_0, then (3.6) holds. Thus the continuity of f at x_0 is tantamount to

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{3.9}$$

In both definitions, after fixing an arbitrary $\varepsilon > 0$, one is asked to find *at least* one positive number δ ('there is a δ') for which (3.6) or (3.8) holds. If either implication holds for a certain δ, it will also hold for every $\delta' < \delta$. The definition does not require to find the biggest possible δ satisfying the implication. With this firmly in mind, testing continuity or verifying a limit can become much simpler.

Returning to the functions f, g, h of the beginning, we can now say that f is continuous at $x_0 = 0$,

$$\lim_{x \to 0} f(x) = 1 = f(0),$$

whereas g, despite having limit 0 for $x \to 0$, is not continuous:

$$\lim_{x \to 0} g(x) = 0 \neq g(0).$$

We shall prove in Example 4.6 i) that h admits a limit for x going to 0, and actually

$$\lim_{x\to 0} h(x) = 1.$$

The functions g and h suggest the following definition.

Definition 3.16 *Let f be defined on a neighbourhood of x_0, excluding the point x_0. If f admits limit $\ell \in \mathbb{R}$ for x approaching x_0, and if a) f is defined in x_0 but $f(x_0) \neq \ell$, or b) f is not defined in x_0, then we say x_0 is a* **(point of) removable discontinuity for** f.

The choice of terminology is justified by the fact that one can *modify* the function at x_0 by *defining* it in x_0, so that to obtain a continuous map at x_0. More precisely, the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0, \\ \ell & \text{if } x = x_0, \end{cases}$$

is such that

$$\lim_{x\to x_0} \tilde{f}(x) = \lim_{x\to x_0} f(x) = \ell = \tilde{f}(x_0),$$

hence it is continuous at x_0.

For the above functions we have $\tilde{g}(x) = x$ in a neighbourhood of the origin, while

$$\tilde{h}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

In the latter case, we have defined the **continuous prolongation** of $y = \dfrac{\sin x}{x}$, by assigning the value that renders it continuous at the origin. From now on when referring to the function $y = \dfrac{\sin x}{x}$, we will always understand it as continuously prolonged in the origin.

Examples 3.17

We show that the main elementary functions are continuous.

i) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ and $x_0 \in \mathbb{R}$ be given. For any $\varepsilon > 0$, $|f(x) - f(x_0)| < \varepsilon$ if and only if $|a|\,|x - x_0| < \varepsilon$. When $a = 0$, the condition holds for any $x \in \mathbb{R}$; if $a \neq 0$ instead, it is equivalent to $|x - x_0| < \dfrac{\varepsilon}{|a|}$, and we can put $\delta = \dfrac{\varepsilon}{|a|}$ in (3.6). The map f is thus continuous at every $x_0 \in \mathbb{R}$.

ii) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous at $x_0 = 2$. We shall prove this fact in two different ways. Given $\varepsilon > 0$, $|f(x) - f(2)| < \varepsilon$, or $|x^2 - 4| < \varepsilon$, means

$$4 - \varepsilon < x^2 < 4 + \varepsilon. \qquad (3.10)$$

We can suppose $\varepsilon \leq 4$ (for if $|f(x) - f(2)| < \varepsilon$ for a certain ε, the same will be true for all $\varepsilon' > \varepsilon$); as we are looking for x in a neighbourhood of 2, we can furthermore assume $x > 0$. Under such assumptions (3.10) yields

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon},$$

hence

$$-(2-\sqrt{4-\varepsilon}) < x-2 < \sqrt{4+\varepsilon}-2. \tag{3.11}$$

This suggests to take $\delta = \min(2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2)$ $(= \sqrt{4+\varepsilon}-2$, easy to verify). If $|x-2| < \delta$, then (3.11) holds, which was equivalent to $|x^2-4| < \varepsilon$. With a few algebraic computations, this furnishes the *greatest* δ for which the inequality $|x^2-4| < \varepsilon$ is true.

We have already said that the largest value of δ is not required by the definitions, so we can also proceed alternatively. Since

$$|x^2-4| = |(x-2)(x+2)| = |x-2|\,|x+2|\,,$$

by restricting x to a neighbourhood of 2 of radius < 1, we will have $-1 < x-2 < 1$, hence $1 < x < 3$. The latter will then give $3 < x+2 = |x+2| < 5$. Thus

$$|x^2-4| < 5|x-2|. \tag{3.12}$$

To obtain $|x^2-4| < \varepsilon$ it will suffice to demand $|x-2| < \frac{\varepsilon}{5}$; since (3.12) holds when $|x-2| < 1$, we can set $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$ and the condition (3.6) will be satisfied. The neighbourhood of radius < 1 was arbitrary: we could have chosen any other sufficiently small neighbourhood and obtain another δ, still respecting the continuity requirement.

Note at last that a similar reasoning tells f is continuous at every $x_0 \in \mathbb{R}$.

iii) We verify that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$ is continuous at every $x_0 \in \mathbb{R}$. We establish first a simple but fundamental inequality.

Lemma 3.18 *For any $x \in \mathbb{R}$,*

$$|\sin x| \leq |x|, \tag{3.13}$$

with equality holding if and only if $x = 0$.

Proof. Let us start assuming $0 < x \leq \frac{\pi}{2}$ and look at the right-angled triangle PHA of Fig. 3.7. The vertical side PH is shorter than the hypotenuse PA, whose length is in turn less than the length of the arc PA (the shortest distance between two points is given by the straight line joining them):

$$\overline{PH} < \overline{PA} < \overset{\frown}{PA}\,.$$

By definition $\overline{PH} = \sin x > 0$, and $\overset{\frown}{PA} = x > 0$ (angles being in radians). Thus (3.13) is true. The case $-\frac{\pi}{2} \leq x < 0$ is treated with the same

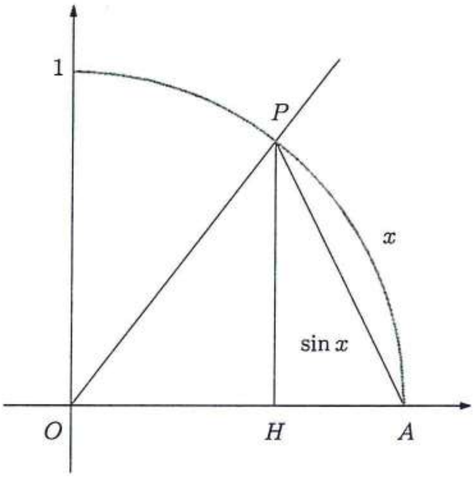


Figure 3.7. $|\sin x| \le |x|$

argument observing $|\sin x| = \sin|x|$ for $0 < |x| \le \frac{\pi}{2}$. At last, when $|x| > \frac{\pi}{2}$ one has $|\sin x| \le 1 < \frac{\pi}{2} < |x|$, ending the proof. □

Thanks to (3.13) we can prove that sine is a continuous function. Recalling formula (2.14),

$$\sin x - \sin x_0 = 2\sin\frac{x-x_0}{2}\cos\frac{x+x_0}{2},$$

(3.13) and the fact that $|\cos t| \le 1$ for all $t \in \mathbb{R}$, imply

$$\begin{aligned}|\sin x - \sin x_0| &= 2\left|\sin\frac{x-x_0}{2}\right|\,\left|\cos\frac{x+x_0}{2}\right| \\ &\le 2\left|\frac{x-x_0}{2}\right|\cdot 1 = |x-x_0|.\end{aligned}$$

Therefore, given an $\varepsilon > 0$, if $|x - x_0| < \varepsilon$ we have $|\sin x - \sin x_0| < \varepsilon$; in other words, condition (3.6) is satisfied by $\delta = \varepsilon$.
Similarly, formula (2.15) allows to prove $g(x) = \cos x$ is continuous at every $x_0 \in \mathbb{R}$. □

Definition 3.19 *Let I be a subset of* dom f. *The function f is called* **continuous on** I (*or* **over** I) *if f is continuous at every point of I.*

We remark that the use of the term 'map' (or 'mapping') is very different from author to author; in some books a map is simply a function (we have adopted this convention), for others the word 'map' automatically assumes continuity, so attention is required when browsing the literature.

The following result is particularly relevant and will be used many times without explicit mention. For its proof, see Appendix A.2.2, p. 436.

Proposition 3.20 *All elementary functions (polynomials, rational functions, powers, trigonometric functions, exponentials and their inverses) are continuous over their entire domains.*

Let us point out that there exists a notion of continuity of a function on a subset of its domain, that is stronger than the one given in Definition 3.19; it is called *uniform continuity*. We refer to Appendix A.3.3, p. 447, for its definition and main properties.

Now back to limits. A function f defined in a neighbourhood of x_0, x_0 excluded, may assume bigger and bigger values as the independent variable x gets closer to x_0. Consider for example the function

$$f(x) = \frac{1}{(x-3)^2}$$

on $\mathbb{R} \setminus \{3\}$, and fix an arbitrarily large real number $A > 0$. Then $f(x) > A$ for all $x \neq x_0$ such that $|x - 3| < \dfrac{1}{\sqrt{A}}$. We would like to say that f *tends to* $+\infty$ for x approaching x_0; the precise definition is as follows.

Definition 3.21 *Let f be defined in a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. The function f* **has limit** $+\infty$ *(or* **tends to** $+\infty$*)* **for** x **approaching** x_0*, in symbols*

$$\lim_{x \to x_0} f(x) = +\infty,$$

if for any $A > 0$ there is a $\delta > 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad 0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > A. \tag{3.14}$$

Otherwise said, for any neighbourhood $I_A(+\infty)$ of $+\infty$ there exists a neighbourhood $I_\delta(x_0)$ di x_0 such that

$$\boxed{\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \setminus \{x_0\} \quad \Rightarrow \quad f(x) \in I_A(+\infty).}$$

The definition of

$$\boxed{\lim_{x \to x_0} f(x) = -\infty}$$

follows by changing $f(x) > A$ to $f(x) < -A$.

One also writes

$$\boxed{\lim_{x \to x_0} f(x) = \infty}$$

to indicate $\lim_{x \to x_0} |f(x)| = +\infty$. For instance the hyperbola $f(x) = \frac{1}{x}$, with graph in Fig. 2.2, *does not admit limit* for $x \to 0$, because on each neighbourhood $I_\delta(0)$ of the origin the function assumes both arbitrarily large positive and negative values together. On the other hand, $|f(x)|$ tends to $+\infty$ when x nears 0. In fact, for fixed $A > 0$

$$\forall x \in \mathbb{R} \setminus \{0\}, \qquad |x| < \frac{1}{A} \quad \Rightarrow \quad \frac{1}{|x|} > A.$$

Hence $\lim_{x \to 0} \frac{1}{x} = \infty$.

3.3.3 One-sided limits; points of discontinuity

The previous example shows that a map may have different limit behaviours at the left and right of a point x_0. The function $f(x) = \dfrac{1}{x}$ grows indefinitely as x takes *positive* values tending to 0; at the same time it becomes smaller as x goes to 0 assuming *negative* values. Consider the graph of the mantissa $y = M(x)$ (see Fig. 2.3, p. 34) on a neighbourhood of $x_0 = 1$ of radius < 1. Then

$$M(x) = \begin{cases} x & \text{if } x < 1, \\ x - 1 & \text{if } x \ge 1. \end{cases}$$

When x approaches 1, M tends to 0 if x takes values > 1 (i.e., at the right of 1), and tends to 1 if x assumes values < 1 (at the left).

The notions of *right-hand limit* and *left-hand limit* (or simply *right limit* and *left limit*) arise from the need to understand these cases. For that, we define **right neighbourhood of x_0 of radius $r > 0$** the bounded half-open interval

$$\boxed{I_r^+(x_0) = [x_0, x_0 + r) = \{x \in \mathbb{R} \ : \ 0 \le x - x_0 < r\}.}$$

The **left neighbourhood of x_0 of radius $r > 0$** will be, similarly,

$$\boxed{I_r^-(x_0) = (x_0 - r, x_0] = \{x \in \mathbb{R} \ : \ 0 \le x_0 - x < r\}.}$$

Substituting the condition $0 < |x - x_0| < \delta$ (i.e., $x \in I_\delta(x_0) \setminus \{x_0\}$) with $0 < x - x_0 < \delta$ (i.e., $x \in I_\delta^+(x_0) \setminus \{x_0\}$) in Definitions 3.15 and 3.21 produces the corresponding definitions for **right limit of f for x tending to x_0**, otherwise said **limit of f for x approaching x_0 from the right** or **limit on the right**; such will be denoted by

$$\lim_{x \to x_0^+} f(x).$$

For a finite limit, this reads as follows.

Definition 3.22 *Let f be defined on a right neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. The function f* **has right limit $\ell \in \mathbb{R}$ for $x \to x_0$**, *if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\forall x \in \operatorname{dom} f, \qquad 0 < x - x_0 < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon.$$

Alternatively, for any neighbourhood $I_\varepsilon(\ell)$ di ℓ there exists a right neighbourhood $I_\delta^+(x_0)$ of x_0 such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I_\delta^+(x_0) \setminus \{x_0\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

The notion of continuity on the right is analogous.

Definition 3.23 *A function f defined on a right neighbourhood of $x_0 \in \mathbb{R}$ is called* **continuous on the right at x_0** *(or* **right-continuous***) if*

$$\lim_{x \to x_0^+} f(x) = f(x_0).$$

If a function is only defined on a right neighbourhood of x_0, right-continuity coincides with the earlier Definition (3.6). The function $f(x) = \sqrt{x}$ for example is defined on $[0, +\infty)$, and is continuous at 0.

Limits of f from the left and left-continuity are completely similar: now one has to use left neighbourhoods of x_0; the left limit shall be denoted by

$$\lim_{x \to x_0^-} f(x).$$

The following easy-to-prove property provides a handy criterion to study limits and continuity.

Proposition 3.24 *Let f be defined in a neighbourhood of $x_0 \in \mathbb{R}$, with the possible exception of x_0. The function f has limit L (finite or infinite) for $x \to x_0$ if and only if the right and left limits of f, for $x \to x_0$, exist and equal L.*

A function f defined in a neighbourhood of x_0 is continuous at x_0 if and only if it is continuous on the right and on the left at x_0.

Returning to the previous examples, it is not hard to see

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty; \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

and

$$\lim_{x\to 1^+} M(x) = 0; \qquad \lim_{x\to 1^-} M(x) = 1.$$

Note $M(1) = 0$, so $\lim\limits_{x\to 1^+} M(x) = M(1)$. All this means the function $M(x)$ is continuous on the right at $x_0 = 1$ (but not left-continuous, hence neither continuous, at $x_0 = 1$).

Definition 3.25 *Let f be defined on a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. If the left and right limits of f for x going to x_0 are different, we say that x_0 is a* **(point of) discontinuity of the first kind** *(or a* **jump point***) for f. The* **gap value** *of f at x_0 is the difference*

$$[f]_{x_0} = \lim_{x\to x_0^+} f(x) - \lim_{x\to x_0^-} f(x).$$

Thus the mantissa has a gap $= -1$ at $x_0 = 1$ and, in general, at each point $x_0 = n \in \mathbb{Z}$.

Also the floor function $y = [x]$ jumps, at each $x_0 = n \in \mathbb{Z}$, with gap $= 1$, for

$$\lim_{x\to n^+} [x] = n; \qquad \lim_{x\to n^-} [x] = n - 1.$$

The sign function $y = \text{sign}\,(x)$ has a jump point at $x_0 = 0$, with gap $= 2$:

$$\lim_{x\to 0^+} \text{sign}\,(x) = 1; \qquad \lim_{x\to 0^-} \text{sign}\,(x) = -1.$$

Definition 3.26 *A discontinuity point which is not removable, nor of the first kind is said* **of the second kind.**

This occurs for instance when f does not admit limit (neither on the left nor on the right) for $x \to x_0$. The function $f(x) = \sin \frac{1}{x}$ has no limit for $x \to 0$ (see Fig. 3.8 and the explanation in Remark 4.19).

3.3.4 Limits of monotone functions

Monotonicity affects the possible limit behaviour of a map, as the following results explain.

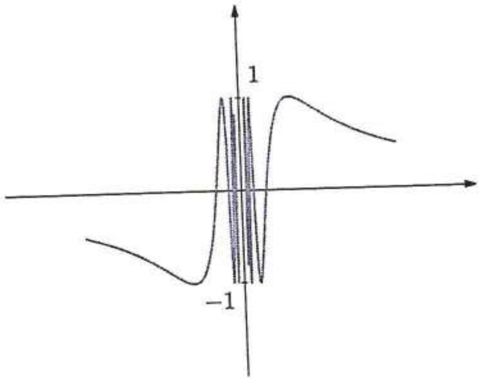


Figure 3.8. Graph of $f(x) = \sin \frac{1}{x}$

Theorem 3.27 *Let f be a monotone function defined on a right neighbourhood $I^+(c)$ of the point c (where c is real or $-\infty$), possibly without the point c itself. Then the right limit for $x \to c$ exists (finite or infinite), and precisely*

$$\lim_{x\to c^+} f(x) = \begin{cases} \inf\{f(x) : x \in I^+(c),\, x > c\} & \text{if } f \text{ is increasing,} \\ \sup\{f(x) : x \in I^+(c),\, x > c\} & \text{if } f \text{ is decreasing.} \end{cases}$$

In the same way, f monotone on a left neighbourhood $I^-(c)\setminus\{c\}$ of c (c real or $+\infty$) satisfies

$$\lim_{x\to c^-} f(x) = \begin{cases} \sup\{f(x) : x \in I^-(c),\, x < c\} & \text{if } f \text{ is increasing,} \\ \inf\{f(x) : x \in I^-(c),\, x < c\} & \text{if } f \text{ is decreasing.} \end{cases}$$

Proof. We shall prove that if f increases in the right neighbourhood $I^+(c)$ of c then

$$\lim_{x\to c^+} f(x) = \inf\{f(x) : x \in I^+(c),\, x > c\}\,.$$

The other cases are similar.
Let $\ell = \inf\{f(x) : x \in I^+(c),\, x > c\} \in \mathbb{R}$. The infimum is characterised, in analogy with (1.7), by:
i) *for all $x \in I^+(c) \setminus \{c\}$, $f(x) \ge \ell$;*
ii) *for any $\varepsilon > 0$, there exists an element $x_\varepsilon \in I^+(c)\setminus\{c\}$ such that $f(x_\varepsilon) < \ell + \varepsilon$.*

By monotonicity we have

$$f(x) \le f(x_\varepsilon)\,, \qquad \forall x \in I^+(c) \setminus \{c\},\, x < x_\varepsilon\,,$$

therefore

$$\ell - \varepsilon < \ell \leq f(x) < \ell + \varepsilon, \qquad \forall x \in I^+(c) \setminus \{c\},\, x < x_\varepsilon\,.$$

So, each $f(x)$ belongs to the neighbourhood of ℓ of radius ε if $x \neq c$ is in the right neighbourhood of c with supremum x_ε. Thus we have

$$\lim_{x \to c^+} f(x) = \ell\,.$$

Let now $\ell = -\infty$; this means that for any $A > 0$ there is an $x_A \in I^+(c) \setminus \{c\}$ such that $f(x_A) < -A$. Using monotonicity again we obtain $f(x) \leq f(x_A) < -A$, $\forall x \in I^+(c) \setminus \{c\}$ and $x < x_A$. Hence $f(x)$ belongs to the neighbourhood of $-\infty$ with supremum $-A$ provided $x \neq c$ is in the right neighbourhood of c of supremum x_A. We conclude

$$\lim_{x \to c^+} f(x) = -\infty\,.$$ □

A straightforward consequence is that a monotone function can have only a discontinuity of the first kind.

Corollary 3.28 *Let f be monotone on a neighbourhood $I(x_0)$ of $x_0 \in \mathbb{R}$. Then the right and left limits for $x \to x_0$ exist and are finite. More precisely,*

i) if f is increasing

$$\lim_{x \to x_0^-} f(x) \leq f(x_0) \leq \lim_{x \to x_0^+} f(x);$$

ii) if f is decreasing

$$\lim_{x \to x_0^-} f(x) \geq f(x_0) \geq \lim_{x \to x_0^+} f(x).$$

Proof. Let f be increasing. Then for all $x \in I(x_0)$ with $x < x_0$, $f(x) \leq f(x_0)$. The above theorem guarantees that

$$\lim_{x \to x_0^-} f(x) = \sup\{f(x) : x \in I(x_0),\, x < x_0\} \leq f(x_0).$$

Similarly, for $x \in I(x_0)$ with $x > x_0$,

$$f(x_0) \leq \inf\{f(x) : x \in I(x_0),\, x > x_0\} = \lim_{x \to x_0^+} f(x),$$

from which *i)* follows. The second implication is alike. □

3.4 Exercises

1. *Using the definition prove that*

a) $\lim_{n\to+\infty} n! = +\infty$

b) $\lim_{n\to+\infty} \frac{n^2}{1-2n} = -\infty$

c) $\lim_{x\to1} (2x^2+3) = 5$

d) $\lim_{x\to2^\pm} \frac{1}{x^2-4} = \pm\infty$

e) $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-1}} = -1$

f) $\lim_{x\to+\infty} \frac{x^2}{1-x} = -\infty$

2. *Let* $f(x) = \text{sign}\,(x^2 - x)$. *Discuss the existence of the limits*

$$\lim_{x\to0} f(x) \qquad \text{and} \qquad \lim_{x\to1} f(x)$$

and study the function's continuity.

3. *Determine the values of the real parameter* α *for which the following maps are continuous on their respective domains:*

a) $f(x) = \begin{cases} \alpha\sin(x+\frac{\pi}{2}) & \text{if } x>0\,, \\ 2x^2+3 & \text{if } x\le 0 \end{cases}$

b) $f(x) = \begin{cases} 3e^{\alpha x-1} & \text{if } x\ge 1\,, \\ x+2 & \text{if } x<1 \end{cases}$

3.4.1 Solutions

1. *Limits:*

a) Let a real number $A>0$ be given; it is sufficient to choose any natural number $n_A \ge A$ and notice that if $n > n_A$ then

$$n! = n(n-1)\cdots 2\cdot 1 \ge n > n_A \ge A.$$

Thus $\lim_{n\to+\infty} n! = +\infty$.

b) Fix a real $A>0$ and note $\frac{n^2}{1-2n} < -A$ is the same as $\frac{n^2}{2n-1} > A$. For $n\ge1$, that means $n^2-2An+A>0$. If we consider a natural number $n_A \ge A+\sqrt{A(A+1)}$, the inequality holds for all $n > n_A$.

c) Fix $\varepsilon>0$ and study the condition $|f(x)-\ell|<\varepsilon$:

$$|2x^2+3-5| = 2|x^2-1| = 2|x-1|\,|x+1| < \varepsilon\,.$$

Without loss of generality we assume x belongs to the neighbourhood of 1 of radius 1, i.e.,