4

Limits and continuity II

The study of limits continues with the discussion of tools that facilitate computations and avoid having to resort to the definition each time. We introduce the notion of indeterminate form, and infer some remarkable limits. The last part of the chapter is devoted to continuous functions on real intervals.

4.1 Theorems on limits

A bit of notation to begin with: the symbol c will denote any of x_0, x_0^+, x_0^-, $+\infty$, $-\infty$, ∞ introduced previously. Correspondingly, $I(c)$ will be a neighbourhood $I_\delta(x_0)$ of $x_0 \in \mathbb{R}$ of radius δ, a right neighbourhood $I_\delta^+(x_0)$, a left neighbourhood $I_\delta^-(x_0)$, a neighbourhood $I_B(+\infty)$ of $+\infty$ with end-point $B > 0$, a neighbourhood $I_B(-\infty)$ of $-\infty$ with end-point $-B$, or a neighbourhood $I_B(\infty) = I_B(-\infty) \cup I_B(+\infty)$ of ∞.

We shall suppose from now on $f, g, h, \ldots$ are functions *defined on a neighbourhood of* c with the point c deleted, unless otherwise stated. In accordance with the meaning of c, the expression $\lim_{x \to c} f(x)$ will stand for the limit of f for $x \to x_0 \in \mathbb{R}$, the right or left limit, the limit for x tending to $+\infty$, $-\infty$, or for $|x| \to +\infty$.

4.1.1 Uniqueness and sign of the limit

We start with the uniqueness of a limit, which justifies having so far said 'the limit of f', in place of 'a limit of f'.

Theorem 4.1 (Uniqueness of the limit) *Suppose f admits (finite or infinite) limit ℓ for $x \to c$. Then f admits no other limit for $x \to c$.*

$$-1 < x - 1 < 1, \quad \text{whence} \quad 0 < x < 2 \quad \text{and} \quad 1 < x + 1 = |x + 1| < 3.$$

Therefore

$$|2x^2 + 3 - 5| < 2 \cdot 3|x - 1| = 6|x - 1|.$$

The expression on the right is $< \varepsilon$ if $|x - 1| < \frac{\varepsilon}{6}$. It will be enough to set $\delta = \min(1, \frac{\varepsilon}{6})$ to prove the claim.

2. Since $x^2 - x > 0$ when $x < 0$ or $x > 1$, the function $f(x)$ is thus defined:

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \text{ and } x > 1, \\ 0 & \text{if } x = 0 \text{ and } x = 1, \\ -1 & \text{if } 0 < x < 1. \end{cases}$$

So f is constant on the intervals $(-\infty, 0)$, $(0, 1)$, $(1, +\infty)$ and

$$\lim_{x \to 0^-} f(x) = 1, \qquad \lim_{x \to 0^+} f(x) = -1,$$
$$\lim_{x \to 1^-} f(x) = -1, \qquad \lim_{x \to 1^+} f(x) = 1.$$

The required limits do not exist. The function is continuous on all $\mathbb{R}$ with the exception of the jump points $x = 0$ and $x = 1$.

3. *Continuity:*

a) The domain of f is $\mathbb{R}$ and the function is continuous for $x \neq 0$, irrespective of α. As for the continuity at $x = 0$, observe that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x^2 + 3) = 3 = f(0),$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \alpha \sin(x + \frac{\pi}{2}) = \alpha.$$

These imply f is continuous also in $x = 0$ if $\alpha = 3$.

b) $\alpha = 1$.

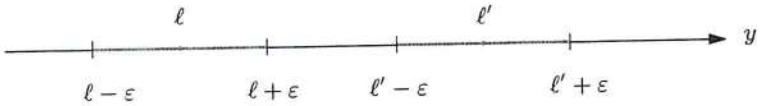

Figure 4.1. The neighbourhoods of ℓ, ℓ' of radius $\varepsilon \leq \frac{1}{2}|\ell - \ell'|$ are disjoint

Proof. We assume there exist two limits $\ell' \neq \ell$ and infer a contradiction. We consider only the case where ℓ and ℓ' are both finite, for the other situations can be easily deduced adapting the same argument. First of all, since $\ell' \neq \ell$ there exist disjoint neighbourhoods $I(\ell)$ of ℓ and $I(\ell')$ of ℓ'

$$I(\ell) \cap I(\ell') = \emptyset. \tag{4.1}$$

To see this fact, it is enough to consider neighbourhoods of radius ε smaller or equal than half the distance of ℓ and ℓ', $\varepsilon \leq \frac{1}{2}|\ell - \ell'|$ (Fig. 4.1).
Taking $I(\ell)$, the hypothesis $\lim\limits_{x \to c} f(x) = \ell$ implies the existence of a neighbourhood $I(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell).$$

Similarly for $I(\ell')$, from $\lim\limits_{x \to c} f(x) = \ell'$ it follows there is $I'(c)$ with

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell').$$

The intersection of $I(c)$ and $I'(c)$ is itself a neighbourhood of c: it contains infinitely many points of the domain of f since we assumed f was defined in a neighbourhood of c (possibly minus c). Therefore if $\bar{x} \in \operatorname{dom} f$ is any point in the intersection, different from c,

$$f(\bar{x}) \in I(\ell) \cap I(\ell'),$$

hence the intervals $I(\ell)$ and $I(\ell')$ do have non-empty intersection, contradicting (4.1). □

The second property we present concerns the sign of a limit around a point c.

Theorem 4.2 *Suppose f admits limit ℓ (finite or infinite) for $x \to c$. If $\ell > 0$ or $\ell = +\infty$, there exists a neighbourhood $I(c)$ of c such that f is strictly positive on $I(c) \setminus \{c\}$. A similar assertion holds when $\ell < 0$ or $\ell = -\infty$.*

Proof. Assume ℓ is finite, positive, and consider the neighbourhood $I_\varepsilon(\ell)$ of ℓ of radius $\varepsilon = \ell/2 > 0$. According to the definition, there is a neighbourhood $I(c)$ of c satisfying

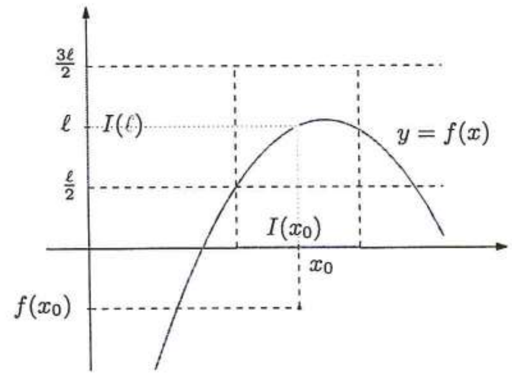

Figure 4.2. Around a limit value, the sign of a map does not change

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

As $I_\varepsilon(\ell) = (\frac{\ell}{2}, \frac{3\ell}{2}) \subset (0, +\infty)$, all values $f(x)$ are positive.
If $\ell = +\infty$ it suffices to take a neighbourhood $I_A(+\infty) = (A, +\infty)$ of $+\infty$ $(A > 0)$ and use the corresponding definition of limit. □

The next result explains in which sense the implication in Theorem 4.2 can be 'almost' reversed.

Corollary 4.3 *Assume f admits limit ℓ (finite or infinite) for x tending to c. If there is a neighbourhood $I(c)$ of c such that $f(x) \geq 0$ in $I(c) \setminus \{c\}$, then $\ell \geq 0$ or $\ell = +\infty$. A similar assertion holds for a 'negative' limit.*

Proof. By contradiction, if $\ell = -\infty$ or $\ell < 0$, Theorem 4.2 would provide a neighbourhood $I'(c)$ of c such that $f(x) < 0$ on $I'(c) \setminus \{c\}$. On the intersection of $I(c)$ and $I'(c)$ we would then simultaneously have $f(x) < 0$ and $f(x) \geq 0$, which is not possible. □

Note that even assuming the stronger inequality $f(x) > 0$ on $I(c)$, we would not be able to exclude ℓ might be zero. For example, the map

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

is strictly positive in every neighbourhood of the origin, yet $\lim_{x \to 0} f(x) = 0$.

4.1.2 Comparison theorems

A few results are known that allow to compare the behaviour of functions, the first of which generalises the above corollary.

Corollary 4.4 (First comparison theorem) *Let a function f have limit ℓ and a function g limit m (ℓ, m finite or not) for $x \to c$. If there is a neighbourhood $I(c)$ of c such that $f(x) \le g(x)$ in $I(c) \setminus \{c\}$, then $\ell \le m$.*

Proof. If $\ell = -\infty$ or $m = +\infty$ there is nothing to prove. Otherwise, consider the map $h(x) = g(x) - f(x)$. By assumption $h(x) \ge 0$ on $I(c) \setminus \{c\}$. Besides, Theorem 4.10 on the algebra of limits guarantees

$$\lim_{x \to c} h(x) = \lim_{x \to c} g(x) - \lim_{x \to c} f(x) = m - \ell.$$

The previous corollary applied to h forces $m - \ell \ge 0$, hence the claim. □

We establish now two useful criteria on the existence of limits based on comparing a given function with others whose limit is known.

Theorem 4.5 (Second comparison theorem – finite case, also known as "**Squeeze rule**") *Let functions f, g and h be given, and assume f and h have the same finite limit for $x \to c$, precisely*

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = \ell.$$

If there is a neighbourhood $I(c)$ of c where the three functions are defined (except possibly at c) and such that

$$f(x) \le g(x) \le h(x), \qquad \forall x \in I(c) \setminus \{c\}, \tag{4.2}$$

then

$$\lim_{x \to c} g(x) = \ell.$$

Proof. We follow the definition of limit for g. Fix a neighbourhood $I_\varepsilon(\ell)$ of ℓ; by the hypothesis $\lim\limits_{x \to c} f(x) = \ell$ we deduce the existence of a neighbourhood $I'(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

The condition $f(x) \in I_\varepsilon(\ell)$ can be written as $|f(x) - \ell| < \varepsilon$, or

$$\ell - \varepsilon < f(x) < \ell + \varepsilon. \tag{4.3}$$

recalling (1.4). Similarly, $\lim\limits_{x \to c} h(x) = \ell$ implies there is a neighbourhood $I''(c)$ of c such that

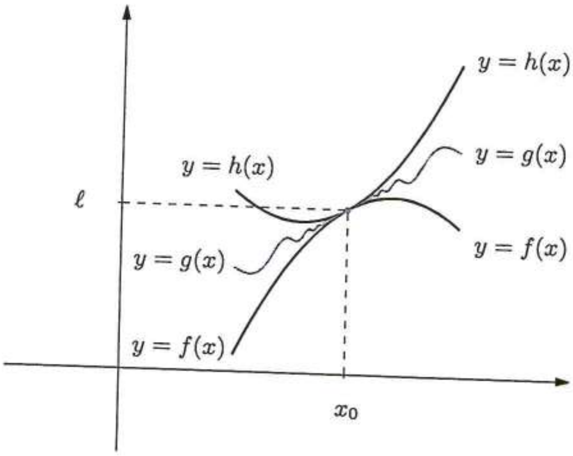

Figure 4.3. The squeeze rule

$$\forall x \in \operatorname{dom} h, \qquad x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad \ell - \varepsilon < h(x) < \ell + \varepsilon. \tag{4.4}$$

Define then $I'''(c) = I(c) \cap I'(c) \cap I''(c)$. On $I'''(c) \setminus \{c\}$ the constraints (4.2), (4.3) and (4.4) all hold, hence in particular

$$x \in I'''(c) \setminus \{c\} \quad \Rightarrow \quad \ell - \varepsilon < f(x) \leq g(x) \leq h(x) < \ell + \varepsilon.$$

This means $g(x) \in I_\varepsilon(\ell)$, concluding the proof. □

Examples 4.6

i) Let us prove the fundamental limit

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.} \tag{4.5}$$

Observe first that $y = \dfrac{\sin x}{x}$ is even, for $\dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x}$. It is thus sufficient to consider a positive x tending to 0, i.e., prove that $\displaystyle\lim_{x \to 0^+} \frac{\sin x}{x} = 1$. Recalling (3.13), for all $x > 0$ we have $\sin x < x$, or $\dfrac{\sin x}{x} < 1$. To find a lower bound, suppose $x < \frac{\pi}{2}$ and consider points on the unit circle: let A have coordinates $(1, 0)$, P coordinates $(\cos x, \sin x)$ and let Q be defined by $(1, \tan x)$ (Fig. 4.4). The circular sector OAP is a proper subset of the triangle OAQ, so

$$\text{Area } OAP < \text{ Area } OAQ.$$

Since

$$\text{Area } OAP = \frac{\overline{OA} \cdot \widehat{AP}}{2} = \frac{x}{2} \qquad \text{and} \qquad \text{Area } OAQ = \frac{\overline{OA} \cdot \overline{AQ}}{2} = \frac{\tan x}{2},$$

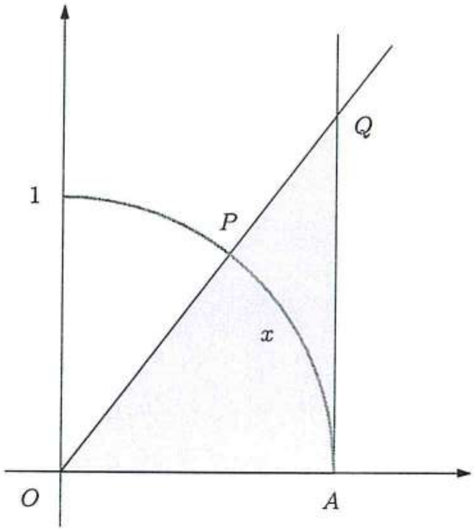

Figure 4.4. The sector OAP is properly contained in OAQ

it follows

$$\frac{x}{2} < \frac{\sin x}{2\cos x}, \qquad \text{i.e.,} \qquad \cos x < \frac{\sin x}{x}.$$

Eventually, on $0 < x < \frac{\pi}{2}$ one has

$$\cos x < \frac{\sin x}{x} < 1.$$

The continuity of the cosine ensures $\lim_{x \to 0+} \cos x = 1$. Now the claim follows from the Second comparison theorem.

ii) We would like to study how the function $g(x) = \frac{\sin x}{x}$ behaves for x tending to $+\infty$. Remember that

$$-1 \le \sin x \le 1 \tag{4.6}$$

for any real x. Dividing by $x > 0$ will not alter the inequalities, so in every neighbourhood $I_A(+\infty)$ of $+\infty$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Now set $f(x) = -\frac{1}{x}$, $h(x) = \frac{1}{x}$ and note $\lim_{x \to +\infty} \frac{1}{x} = 0$. By the previous theorem

$$\lim_{x \to +\infty} \frac{\sin x}{x} = 0.$$

□

The latter example is part of a more general result which we state next (and both are consequences of Theorem 4.5).

Corollary 4.7 *Let f be a* **bounded** *function around c, i.e., there exist a neighbourhood $I(c)$ and a constant $C > 0$ such that*

$$|f(x)| \leq C, \qquad \forall x \in I(c) \setminus \{c\}. \tag{4.7}$$

Let g be such that

$$\lim_{x \to c} g(x) = 0.$$

Then it follows

$$\lim_{x \to c} f(x)g(x) = 0.$$

Proof. By definition $\lim_{x \to c} g(x) = 0$ if and only if $\lim_{x \to c} |g(x)| = 0$, and (4.7) implies

$$0 \leq |f(x)g(x)| \leq C|g(x)|, \qquad \forall x \in I(c) \setminus \{c\}.$$

The claim follows by applying Theorem 4.5. □

Theorem 4.8 (Second comparison theorem – infinite case) *Let f, g be given functions and*

$$\lim_{x \to c} f(x) = +\infty.$$

If there exists a neighbourhood $I(c)$ of c, where both functions are defined (except possibly at c), such that

$$f(x) \leq g(x), \qquad \forall x \in I(c) \setminus \{c\}, \tag{4.8}$$

then

$$\lim_{x \to c} g(x) = +\infty.$$

A result of the same kind for f holds when the limit of g is $-\infty$.

Proof. The proof is, with the necessary changes, like that of Theorem 4.5, hence left to the reader. □

Example 4.9

Compute the limit of $g(x) = x + \sin x$ when $x \to +\infty$. Using (4.6) we have

$$x - 1 \leq x + \sin x, \qquad \forall x \in \mathbb{R}.$$

Set $f(x) = x - 1$; since $\lim_{x \to +\infty} f(x) = +\infty$, the theorem tells us

$$\lim_{x \to +\infty} (x + \sin x) = +\infty.$$

□

4.1.3 Algebra of limits. Indeterminate forms of algebraic type

This section is devoted to the interaction of limits with the algebraic operations of sum, difference, product and quotient of functions.

First though, we must extend arithmetic operations to treat the symbols $+\infty$ and $-\infty$. Let us set:

$$
\begin{array}{ll}
+\infty + s = +\infty & (\text{if } s \in \mathbb{R} \text{ or } s = +\infty) \\
-\infty + s = -\infty & (\text{if } s \in \mathbb{R} \text{ or } s = -\infty) \\
\pm\infty \cdot s = \pm\infty & (\text{if } s > 0 \text{ or } s = +\infty) \\
\pm\infty \cdot s = \mp\infty & (\text{if } s < 0 \text{ or } s = -\infty) \\
\dfrac{\pm\infty}{s} = \pm\infty & (\text{if } s > 0) \\
\dfrac{\pm\infty}{s} = \mp\infty & (\text{if } s < 0) \\
\dfrac{s}{0} = \infty & (\text{if } s \in \mathbb{R} \setminus \{0\} \text{ or } s = \pm\infty) \\
\dfrac{s}{\pm\infty} = 0 & (\text{if } s \in \mathbb{R})
\end{array}
$$

Instead, the following expressions are *not* defined

$$
\pm\infty + (\mp\infty), \qquad \pm\infty - (\pm\infty), \qquad \pm\infty \cdot 0, \qquad \frac{\pm\infty}{\pm\infty}, \qquad \frac{0}{0}.
$$

A result of the foremost importance comes next.

Theorem 4.10 *Suppose f admits limit ℓ (finite or infinite) and g admits limit m (finite or infinite) for $x \to c$. Then*

$$
\lim_{x\to c} \big(f(x) \pm g(x)\big) = \ell \pm m,
$$

$$
\lim_{x\to c} \big(f(x)\, g(x)\big) = \ell\, m,
$$

$$
\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\ell}{m},
$$

provided the right-hand-side expressions make sense. (In the last case one assumes $g(x) \neq 0$ on some $I(c)\setminus\{c\}$.)

Proof. We shall prove two relations only, referring the reader to Appendix A.2.1, p. 433, for the ones left behind. The first we concentrate upon is

$$\lim_{x \to c} (f(x) + g(x)) = \ell + m$$

when ℓ and m are finite. Fix $\varepsilon > 0$, and consider the neighbourhood of ℓ of radius $\varepsilon/2$. By assumption there is a neighbourhood $I'(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon/2.$$

For the same reason there is also an $I''(c)$ with

$$\forall x \in \operatorname{dom} g, \qquad x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon/2.$$

Put $I(c) = I'(c) \cap I''(c)$. Then if $x \in \operatorname{dom} f \cap \operatorname{dom} g$ belongs to $I(c) \setminus \{c\}$, both inequalities hold; the triangle inequality (1.1) yields

$$\begin{aligned}|(f(x) + g(x)) - (\ell + m)| &= |(f(x) - \ell) + (g(x) - m)| \\ &\leq |f(x) - \ell| + |g(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

proving the assertion.

The second relation is

$$\lim_{x \to c} (f(x)\, g(x)) = +\infty$$

with $\ell = +\infty$ and $m > 0$ finite. For a given real $A > 0$, consider the neighbourhood of $+\infty$ with end-point $B = 2A/m > 0$. We know there is a neighbourhood $I'(c)$ such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) > B.$$

On the other hand, considering the neighbourhood of m of radius $m/2$, there exists an $I''(c)$ such that

$$\forall x \in \operatorname{dom} g, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < m/2,$$

i.e., $m/2 < g(x) < 3m/2$. Set $I(c) = I'(c) \cap I''(c)$. If $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is in $I(c) \setminus \{c\}$, the previous relations will be both fulfilled, whence

$$f(x)\, g(x) > f(x)\, \frac{m}{2} > B\frac{m}{2} = A.$$

□

Corollary 4.11 *If f and g are continuous maps at a point $x_0 \in \mathbb{R}$, then also $f(x) \pm g(x)$, $f(x)\, g(x)$ and $\dfrac{f(x)}{g(x)}$ (provided $g(x_0) \neq 0$) are continuous at x_0.*

Proof. The condition that f and g are continuous at x_0 is equivalent to $\lim\limits_{x\to x_0} f(x) = f(x_0)$ and $\lim\limits_{x\to x_0} g(x) = g(x_0)$ (recall (3.9)). The previous theorem allows to conclude. □

Corollary 4.12 *Rational functions are continuous on their domain. In particular, polynomials are continuous on* $\mathbb{R}$.

Proof. We verified in Example 3.17, part i), that the constants $y = a$ and the linear function $y = x$ are continuous on $\mathbb{R}$. Consequently, maps like $y = ax^n$ ($n \in \mathbb{N}$) are continuous. But then so are polynomials, being sums of the latter. Rational functions, as quotients of polynomials, inherit the property wherever the denominator does not vanish. □

Examples 4.13

i) Calculate

$$\lim_{x\to 0} \frac{2x - 3\cos x}{5 + x\sin x} = \ell.$$

The continuity of numerator and denominator descends from algebraic operations on continuous maps, and the denominator is not zero at $x = 0$. The substitution of 0 to x produces $\ell = -3/5$.

ii) Discuss the limit behaviour of $y = \tan x$ when $x \to \frac{\pi}{2}$. Since

$$\lim_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1 \qquad \text{and} \qquad \lim_{x\to\frac{\pi}{2}} \cos x = \cos\frac{\pi}{2} = 0,$$

the above theorem tells

$$\lim_{x\to\frac{\pi}{2}} \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} = \frac{1}{0} = \infty.$$

But one can be more precise by looking at the sign of the tangent around $\frac{\pi}{2}$. Since $\sin x > 0$ in a neighbourhood of $\frac{\pi}{2}$, while $\cos x > 0$ (< 0) in a left (resp. right) neighbourhood of $\frac{\pi}{2}$, it follows

$$\lim_{x\to\frac{\pi}{2}^{\pm}} \tan x = \mp\infty.$$

iii) Let $R(x) = \dfrac{P(x)}{Q(x)}$ be rational and reduced, meaning the polynomials P, Q have no common factor. Call $x_0 \in \mathbb{R}$ a zero of Q, i.e., a point such that $Q(x_0) = 0$. Clearly $P(x_0) \neq 0$, otherwise P and Q would be both divisible by $(x - x_0)$. Then

$$\lim_{x\to x_0} R(x) = \infty$$

follows. In this case too, the sign of $R(x)$ around of x_0 retains some information. For instance, $y = \dfrac{x^2 - 3x + 1}{x^2 - x}$ is positive on a left neighbourhood of $x_0 = 1$ and negative on a right neighbourhood, so

$$\lim_{x\to 1^\pm} \frac{x^2-3x+1}{x^2-x} = \mp\infty.$$

In contrast, the function $y = \dfrac{x-2}{x^2-2x+1}$ is negative in a whole neighbourhood of $x_0 = 1$, hence

$$\lim_{x\to 1} \frac{x-2}{x^2-2x+1} = -\infty. \qquad \square$$

Theorem 4.10 gives no indication about the limit behaviour of an algebraic expression in three cases, listed below. The expressions in question are called **indeterminate forms** of algebraic type.

i) Consider $f(x)+g(x)$ (resp. $f(x)-g(x)$) when both f, g tend to ∞ with different (resp. same) signs. This gives rise to the indeterminate form denoted by the symbol

$$\infty - \infty.$$

ii) The product $f(x)\,g(x)$, when one function tends to ∞ and the other to 0, is the indeterminate form with symbol

$$\infty \cdot 0.$$

iii) Relatively to $\dfrac{f(x)}{g(x)}$, in case both functions tend to ∞ or 0, the indeterminate forms are denoted with

$$\frac{\infty}{\infty} \qquad \text{or} \qquad \frac{0}{0}.$$

In presence of an indeterminate form, the limit behaviour cannot be told a priori, and there are examples for each possible limit: infinite, finite non-zero, zero, even non-existing limit. Every indeterminate form should be treated singularly and requires often a lot of attention.

Later we shall find the actual limit behaviour of many important indeterminate forms. With those and this section's theorems we will discuss more complicated indeterminate forms. Additional tools to analyse this behaviour will be provided further on: they are the local comparison of functions by means of the Landau symbols (Sect. 5.1), de l'Hôpital's Theorem (Sect. 6.11), the Taylor expansion (Sect. 7.1).

Examples 4.14

i) Let x tend to $+\infty$ and define functions $f_1(x) = x+x^2$, $f_2(x) = x+1$, $f_3(x) = x+\frac{1}{x}$, $f_4(x) = x+\sin x$. Set $g(x) = x$. Using Theorem 4.10, or Example 4.9, one verifies easily that all maps tend to $+\infty$. One has

$$\lim_{x\to+\infty} [f_1(x)-g(x)] = \lim_{x\to+\infty} x^2 = +\infty,$$

$$\lim_{x\to+\infty} [f_2(x)-g(x)] = \lim_{x\to+\infty} 1 = 1,$$

$$\lim_{x\to+\infty} [f_3(x)-g(x)] = \lim_{x\to+\infty} \frac{1}{x} = 0,$$

whereas the limit of $f_4(x) - g(x) = \sin x$ does not exist: the function $\sin x$ is periodic and assumes each value between -1 and 1 infinitely many times as $x \to +\infty$.

ii) Consider now $x \to 0$. Let $f_1(x) = x^3$, $f_2(x) = x^2$, $f_3(x) = x$, $f_4(x) = x^2 \sin \frac{1}{x}$, and $g(x) = x^2$. All functions converge to 0 (for f_4 apply Corollary 4.7). Now

$$\lim_{x\to 0} \frac{f_1(x)}{g(x)} = \lim_{x\to 0} x = 0,$$
$$\lim_{x\to 0} \frac{f_2(x)}{g(x)} = \lim_{x\to 0} 1 = 1,$$
$$\lim_{x\to 0} \frac{f_3(x)}{g(x)} = \lim_{x\to 0} \frac{1}{x}, = \infty,$$

but $\dfrac{f_4(x)}{g(x)} = \sin \dfrac{1}{x}$ does not admit limit for $x \to 0$ (Remark 4.19 furnishes a proof of this).

iii) Let us consider a polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0 \qquad (a_n \neq 0)$$

for $x \to \pm\infty$. A function of this sort can give rise to an indeterminate form $\infty - \infty$ according to the coefficients' signs and the degree of the monomials involved. The problem is sorted by factoring out the leading term (monomial of maximal degree) x^n

$$P(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

The part in brackets converges to a_n when $x \to \pm\infty$, so

$$\boxed{\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n = \infty}$$

The sign of the limit is easily found. For instance,

$$\lim_{x\to-\infty} (-5x^3 + 2x^2 + 7) = \lim_{x\to-\infty} (-5x^3) = +\infty.$$

Take now a reduced rational function

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} \qquad (a_n,\, b_m \neq 0,\ m > 0).$$

When $x \to \pm\infty$, an indeterminate form $\frac{\infty}{\infty}$ arises. With the same technique as before,

$$\boxed{\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m} = \begin{cases} \infty & \text{if } n > m\,, \\ \dfrac{a_n}{b_m} & \text{if } n = m\,, \\ 0 & \text{if } n < m\,. \end{cases}}$$

For example:

$$\lim_{x\to+\infty} \frac{3x^3-2x+1}{x-x^2} = \lim_{x\to+\infty} \frac{3x^3}{-x^2} = -\infty\,,$$

$$\lim_{x\to-\infty} \frac{-4x^5+2x^3-7}{8x^5-x^4+5x} = \lim_{x\to-\infty} \frac{-4x^5}{8x^5} = -\frac{1}{2}\,,$$

$$\lim_{x\to-\infty} \frac{6x^2-x+5}{-x^3+9} = \lim_{x\to-\infty} \frac{6x^2}{-x^3} = 0.$$

iv) The function $y = \dfrac{\sin x}{x}$ becomes indeterminate $\frac{0}{0}$ for $x \to 0$; we proved in part i), Examples 4.6 that y converges to 1. From this, we can deduce the behaviour of $y = \dfrac{1-\cos x}{x^2}$ as $x \to 0$, another indeterminate form of the type $\frac{0}{0}$. In fact,

$$\lim_{x\to0} \frac{1-\cos x}{x^2} = \lim_{x\to0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to0} \frac{1-\cos^2 x}{x^2} \cdot \lim_{x\to0} \frac{1}{1+\cos x}.$$

The fundamental trigonometric equation $\cos^2 x + \sin^2 x = 1$ together with Theorem 4.10 gives

$$\lim_{x\to0} \frac{\sin^2 x}{x^2} = \lim_{x\to0} \left(\frac{\sin x}{x}\right)^2 = \left(\lim_{x\to0} \frac{\sin x}{x}\right)^2 = 1.$$

The same theorem tells also that the second limit is $\frac{1}{2}$, so we conclude

$$\boxed{\lim_{x\to0} \frac{1-\cos x}{x^2} = \frac{1}{2}.}$$

□

With these examples we have taken the chance to look at the behaviour of elementary functions at the boundary points of their domains. For completeness we gather the most significant limits relative to the elementary functions of Sect. 2.6, their proofs may be found in Appendix A.2.2, p. 435.

$\lim_{x\to+\infty} x^\alpha = +\infty\,,$	$\lim_{x\to0^+} x^\alpha = 0$	$\alpha > 0$
$\lim_{x\to+\infty} x^\alpha = 0\,,$	$\lim_{x\to0^+} x^\alpha = +\infty$	$\alpha < 0$
$\lim_{x\to\pm\infty} \dfrac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$	$= \dfrac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$	
$\lim_{x\to+\infty} a^x = +\infty\,,$	$\lim_{x\to-\infty} a^x = 0$	$a > 1$
$\lim_{x\to+\infty} a^x = 0\,,$	$\lim_{x\to-\infty} a^x = +\infty$	$a < 1$
$\lim_{x\to+\infty} \log_a x = +\infty\,,$	$\lim_{x\to0^+} \log_a x = -\infty$	$a > 1$
$\lim_{x\to+\infty} \log_a x = -\infty\,,$	$\lim_{x\to0^+} \log_a x = +\infty$	$a < 1$

$$\lim_{x\to\pm\infty} \sin x\,, \qquad \lim_{x\to\pm\infty} \cos x\,, \qquad \lim_{x\to\pm\infty} \tan x \qquad \text{do not exist}$$

$$\lim_{x\to(\frac{\pi}{2}+k\pi)^{\pm}} \tan x = \mp\infty\,, \quad \forall k \in \mathbb{Z}$$

$$\lim_{x\to\pm1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

4.1.4 Substitution theorem

The so-called Substitution theorem is important in itself for theoretical reasons, besides providing a very useful method to compute limits.

Theorem 4.15 *Suppose a map f admits limit*

$$\lim_{x\to c} f(x) = \ell, \tag{4.9}$$

finite or not. Let g be defined on a neighbourhood of ℓ (excluding possibly the point ℓ) and such that

i) if $\ell \in \mathbb{R}$, g is continuous at ℓ;

ii) if $\ell = +\infty$ or $\ell = -\infty$, the limit $\lim\limits_{y\to\ell} g(y)$ exists, finite or not.

Then the composition $g \circ f$ admits limit for $x \to c$ and

$$\lim_{x\to c} g(f(x)) = \lim_{y\to\ell} g(y). \tag{4.10}$$

Proof. Set $m = \lim\limits_{y\to\ell} g(y)$ (noting that under *i)*, $m = g(\ell)$). Given any neighbourhood $I(m)$ of m, by *i)* or *ii)* there will be a neighbourhood $I(\ell)$ of ℓ such that

$$\forall y \in \operatorname{dom} g, \qquad y \in I(\ell) \quad \Rightarrow \quad g(y) \in I(m).$$

Note that in case *i)* we can use $I(\ell)$ instead of $I(\ell) \setminus \{\ell\}$ because g is continuous at ℓ (recall (3.7)), while ℓ does not belong to $I(\ell)$ for case *ii)*. With such $I(\ell)$, assumption (4.9) implies the existence of a neighbourhood $I(c)$ of c with

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell).$$

Since $x \in \operatorname{dom} g \circ f$ means $x \in \operatorname{dom} f$ plus $y = f(x) \in \operatorname{dom} g$, the previous two implications now give

$$\forall x \in \operatorname{dom} g \circ f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad g(f(x)) \in I(m).$$

But $I(m)$ was arbitrary, so

$$\lim_{x \to c} g(f(x)) = m. \qquad \square$$

Remark 4.16 An alternative condition that yields the same conclusion is the following:

i') if $\ell \in \mathbb{R}$, there is a neighbourhood $I(c)$ of c where $f(x) \neq \ell$ for all $x \neq c$, and the limit $\lim\limits_{y \to \ell} g(y)$ exists, finite or infinite.

The proof is analogous.

$\square$

In case $\ell \in \mathbb{R}$ and g is continuous at ℓ (case *i)*), then $\lim\limits_{y \to \ell} g(y) = g(\ell)$, so (4.10) reads

$$\boxed{\lim_{x \to c} g(f(x)) = g(\lim_{x \to c} f(x)).} \qquad (4.11)$$

An imprecise but effective way to put (4.11) into words is to say that a continuous function *commutes* (exchanges places) with the symbol of limit.

Theorem 4.15 implies that continuity is inherited by composite functions, as we discuss hereby.

Corollary 4.17 *Let f be continuous at x_0, and define $y_0 = f(x_0)$. Let furthermore g be defined around y_0 and continuous at y_0. Then the composite $g \circ f$ is continuous at x_0.*

Proof. From (4.11)

$$\lim_{x \to x_0} (g \circ f)(x) = g(\lim_{x \to x_0} f(x)) = g(f(x_0)) = (g \circ f)(x_0),$$

which is equivalent to the claim.

$\square$

A few practical examples will help us understand how the Substitution theorem and its corollary are employed.

Examples 4.18

i) The map $h(x) = \sin(x^2)$ is continuous on $\mathbb{R}$, being the composition of the continuous functions $f(x) = x^2$ and $g(y) = \sin y$.

ii) Let us determine

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2}.$$

Set $f(x) = x^2$ and

$$g(y) = \begin{cases} \dfrac{\sin y}{y} & \text{if } y \neq 0, \\ 1 & \text{if } y = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 0$, and we know that g is continuous at the origin. Thus

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

iii) We study the behaviour of $h(x) = \arctan\left(\dfrac{1}{x-1}\right)$ around the point 1. Defining $f(x) = \dfrac{1}{x-1}$, we have $\lim_{x\to 1^\pm} f(x) = \pm\infty$. If we call $g(y) = \arctan y$, $\lim_{y\to\pm\infty} g(y) = \pm\dfrac{\pi}{2}$ (see the Table on page 101). Therefore

$$\lim_{x\to 1^\pm} \arctan\left(\frac{1}{x-1}\right) = \lim_{y\to\pm\infty} g(y) = \pm\frac{\pi}{2}.$$

iv) Determine

$$\lim_{x\to+\infty} \log\sin\frac{1}{x}.$$

Setting $f(x) = \sin\frac{1}{x}$ has the effect that $\ell = \lim_{x\to+\infty} f(x) = 0$. Note that $f(x) > 0$ for all $x > \frac{1}{\pi}$. With $g(y) = \log y$ we have $\lim_{y\to 0^+} g(y) = -\infty$, so Remark 4.16 yields

$$\lim_{x\to+\infty} \log\sin\frac{1}{x} = \lim_{y\to 0^+} g(y) = -\infty.$$

□

Remark 4.19 Theorem 4.15 extends easily to cover the case where the role of f is played by a sequence $a : n \mapsto a_n$ with limit

$$\lim_{n\to\infty} a_n = \ell.$$

Namely, under the same assumptions on g,

$$\lim_{n\to\infty} g(a_n) = \lim_{y\to\ell} g(y).$$

This result is often used to disprove the existence of a limit, in that it provides a **Criterion of non-existence for limits**: *if two sequences $a : n \mapsto a_n$, $b : n \mapsto b_n$ have the same limit ℓ and*

$$\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n),$$

then g does not admit limit when its argument tends to ℓ.

For example we can prove, with the aid of the criterion, that $y = \sin x$ has no limit when $x \to +\infty$: define the sequences $a_n = 2n\pi$ and $b_n = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} 0 = 0, \qquad \text{and at the same time} \qquad \lim_{n\to\infty} \sin b_n = \lim_{n\to\infty} 1 = 1.$$

Similarly, the function $y = \sin \frac{1}{x}$ has neither left nor right limit for $x \to 0$. □

4.2 More fundamental limits. Indeterminate forms of exponential type

Consider the paramount limit (3.3). Instead of the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$, we look now at the function of real variable

$$h(x) = \left(1 + \frac{1}{x}\right)^x.$$

It is defined when $1 + \frac{1}{x} > 0$, hence on $(-\infty, -1) \cup (0, +\infty)$. The following result states that h and the sequence resemble each other closely when x tends to infinity. Its proof is given in Appendix A.2.3, p. 439.

Property 4.20 *The following limit holds*

$$\lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{e}.$$

By manipulating this formula we achieve a series of new fundamental limits. The substitution $y = \frac{x}{a}$, with $a \neq 0$, gives

$$\lim_{x\to\pm\infty} \left(1 + \frac{a}{x}\right)^x = \lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^{ay} = \left[\lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^y\right]^a = \mathrm{e}^a.$$

In terms of the variable $y = \frac{1}{x}$ then,

$$\lim_{x\to 0} (1 + x)^{1/x} = \lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^y = \mathrm{e}.$$

The continuity of the logarithm together with (4.11) furnish

$$\lim_{x\to 0} \frac{\log_a(1 + x)}{x} = \lim_{x\to 0} \log_a (1 + x)^{1/x} = \log_a \lim_{x\to 0} (1 + x)^{1/x} = \log_a \mathrm{e} = \frac{1}{\log a}$$

for any $a > 0$. In particular, taking $a = \mathrm{e}$:

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1.$$

Note by the way $a^x - 1 = y$ is equivalent to $x = \log_a(1+y)$, and $y \to 0$ if $x \to 0$. With this substitution,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log_a(1+y)} = \left[\lim_{y \to 0} \frac{\log_a(1+y)}{y} \right]^{-1} = \log a. \tag{4.12}$$

Taking $a = \mathrm{e}$ produces

$$\lim_{x \to 0} \frac{\mathrm{e}^x - 1}{x} = 1.$$

Eventually, let us set $1 + x = \mathrm{e}^y$. Since $y \to 0$ when $x \to 0$,

$$\begin{aligned} \lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} &= \lim_{y \to 0} \frac{\mathrm{e}^{\alpha y} - 1}{\mathrm{e}^y - 1} = \lim_{y \to 0} \frac{\mathrm{e}^{\alpha y} - 1}{y} \frac{y}{\mathrm{e}^y - 1} \\ &= \lim_{y \to 0} \frac{(\mathrm{e}^\alpha)^y - 1}{y} \lim_{y \to 0} \frac{y}{\mathrm{e}^y - 1} = \log \mathrm{e}^\alpha = \alpha \end{aligned} \tag{4.13}$$

for any $\alpha \in \mathbb{R}$.

For the reader's conveniency, all fundamental limits found so far are gathered below.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x \to \pm\infty} \left(1 + \frac{a}{x}\right)^x = \mathrm{e}^a \qquad (a \in \mathbb{R})$$

$$\lim_{x \to 0} (1+x)^{1/x} = \mathrm{e}$$

$$\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a} \ (a > 0); \text{ in particular, } \lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a \quad (a > 0); \qquad \text{in particular, } \lim_{x \to 0} \frac{\mathrm{e}^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad (\alpha \in \mathbb{R}).$$

Let us return to the map $h(x) = \left(1 + \frac{1}{x}\right)^x$. By setting $f(x) = \left(1 + \frac{1}{x}\right)$ and $g(x) = x$, we can write

$$h(x) = [f(x)]^{g(x)}.$$

In general such an expression may give rise to **indeterminate forms** for x tending to a certain c. Suppose f, g are functions defined in a neighbourhood of c, except possibly at c, and that they admit limit for $x \to c$. Assume moreover $f(x) > 0$ around c, so that h is well defined in a neighbourhood of c (except possibly at c). To understand h it is convenient to use the identity

$$f(x) = \mathrm{e}^{\log f(x)}.$$

From this in fact we obtain

$$h(x) = \mathrm{e}^{g(x) \log f(x)}.$$

By continuity of the exponential and (4.11), we have

$$\boxed{\lim_{x \to c} [f(x)]^{g(x)} = \exp\left(\lim_{x \to c} [g(x) \log f(x)]\right).}$$

In other words, $h(x)$ can be studied by looking at the exponent $g(x) \log f(x)$. An indeterminate form of the latter will thus develop an **indeterminate form** of exponential type for $h(x)$. Namely, we might find ourselves in one of these situations:

i) g tends to ∞ and f to 1 (so $\log f$ tends to 0): the exponent is an indeterminate form of type $\infty \cdot 0$, whence we say that h presents an indeterminate form of type

$$1^\infty.$$

ii) g and f both tend to 0 (so $\log f$ tends to $-\infty$): once again the exponent is of type $\infty \cdot 0$, and the function h is said to have an indeterminate form of type

$$0^0.$$

iii) g tends to 0 and f tends to $+\infty$ ($\log f \to +\infty$): the exponent is of type $\infty \cdot 0$, and h becomes indeterminate of type

$$\infty^0.$$

Examples 4.21

i) The map $h(x) = \left(1 + \frac{1}{x}\right)^x$ is an indeterminate form of type 1^∞ when $x \to \pm\infty$, whose limit equals e.

ii) The function $h(x) = x^x$, for $x \to 0^+$, is an indeterminate form of type 0^0. We shall prove in Chap. 6 that $\lim_{x \to 0^+} x \log x = 0$, therefore $\lim_{x \to 0^+} h(x) = 1$.

iii) The function $h(x) = x^{1/x}$ is for $x \to +\infty$ an indeterminate form of type ∞^0. Substituting $y = \frac{1}{x}$, and recalling that $\log \frac{1}{y} = -\log y$, we obtain $\lim_{x\to+\infty} \frac{\log x}{x} = -\lim_{y\to 0^+} y \log y = 0$, hence $\lim_{x\to+\infty} h(x) = 1$. □

When dealing with $h(x) = [f(x)]^{g(x)}$, a rather common mistake – with tragic consequences – is to calculate first the limit of f and/or g, substitute the map with this value and compute the limit of the expression thus obtained. This is to emphasize that **it might be incorrect** to calculate the limit for $x \to c$ of the indeterminate form $h(x) = [f(x)]^{g(x)}$ by finding first

$$m = \lim_{x\to c} g(x), \qquad \text{and from this proceed to} \quad \lim_{x\to c} [f(x)]^m.$$

Equally incorrect might be to determine

$$\lim_{x\to c} \ell^{\,g(x)}, \qquad \text{already knowing} \quad \ell = \lim_{x\to c} f(x).$$

For example, suppose we are asked to find the limit of $h(x) = \left(1 + \frac{1}{x}\right)^x$ for $x \to \pm\infty$; we might think of finding first $\ell = \lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right) = 1$ and from this $\lim_{x\to\pm\infty} 1^x = \lim_{x\to\pm\infty} 1 = 1$. This would lead us to believe, wrongly, that h converges to 1, in spite of the fact the correct limit is e.

4.3 Global features of continuous maps

Hitherto the focus has been on several local properties of functions, whether in the neighbourhood of a real point or a point at infinity, and limits have been discussed in that respect. Now we turn our attention to continuous functions defined on a real interval, and establish properties of global nature, i.e., those relative to the behaviour on the entire domain.

Let us start with a plain definition.

Definition 4.22 *A* **zero** *of a real-valued function f is a point $x_0 \in \operatorname{dom} f$ at which the function vanishes.*

For instance, the zeroes of $y = \sin x$ are the multiples of π, i.e., the elements of the set $\{m\pi \mid m \in \mathbb{Z}\}$.

The problem of solving an equation like

$$f(x) = 0$$

is equivalent to determining the zeroes of the function $y = f(x)$. That is why it becomes crucial to have methods, both analytical and numerical, that allow to find the zeroes of a function, or at least their approximate position.

A simple condition to have a zero inside an interval goes as follows.

Theorem 4.23 (Existence of zeroes) *Let f be a continuous map on a closed, bounded interval $[a,b]$. If $f(a)f(b) < 0$, i.e., if the images of the end-points under f have different signs, f admits a zero within the open interval (a,b).*
If moreover f is strictly monotone on $[a,b]$, the zero is unique.

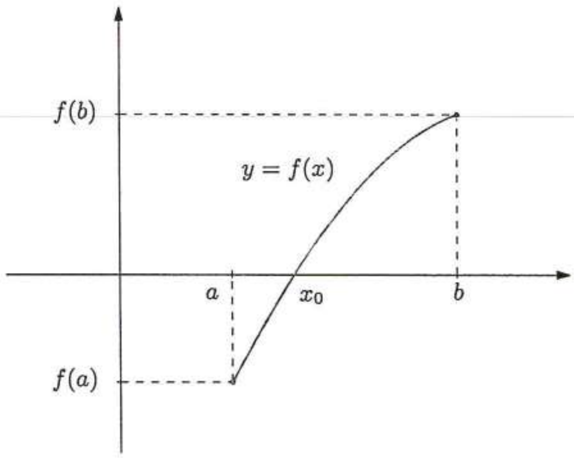

Figure 4.5. Theorem of existence of zeroes

Proof. Throughout the proof we shall use properties of sequences, for which we refer to the following Sect. 5.4. Assuming $f(a) < 0 < f(b)$ is not restrictive. Define $a_0 = a$, $b_0 = b$ and let $c_0 = \frac{a_0+b_0}{2}$ be the middle point of the interval $[a_0, b_0]$. There are three possibilities for $f(c_0)$. If $f(c_0) = 0$, the point $x_0 = c_0$ is a zero and the proof ends. If $f(c_0) > 0$, we set $a_1 = a_0$ and $b_1 = c_0$, so to consider the left half of the original interval. If $f(c_0) < 0$, let $a_1 = c_0$, $b_1 = b_0$ and take the right half of $[a_0, b_0]$ this time. In either case we have generated a sub-interval $[a_1, b_1] \subset [a_0, b_0]$ such that

$$f(a_1) < 0 < f(b_1) \quad \text{and} \quad b_1 - a_1 = \frac{b_0 - a_0}{2}.$$

Repeating the procedure we either reach a zero of f after a finite number of steps, or we build a sequence of nested intervals $[a_n, b_n]$ satisfying:

$$[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \supset \ldots,$$
$$f(a_n) < 0 < f(b_n) \quad \text{and} \quad b_n - a_n = \frac{b_0 - a_0}{2^n}$$

(the rigorous proof of the existence of such a sequence relies on the Principle of Induction; details are provided in Appendix A.1, p. 429). In this second situation, we claim that there is a unique point x_0 belonging to every interval of the sequence, and this point is a zero of f. For this, observe that the sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$$a_0 \leq a_1 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_1 \leq b_0.$$

Therefore $\{a_n\}$ is monotone increasing and bounded, while $\{b_n\}$ is monotone decreasing and bounded. By Theorem 3.9 there exist $x_0^-, x_0^+ \in [a, b]$ such that

$$\lim_{n\to\infty} a_n = x_0^- \quad \text{and} \quad \lim_{n\to\infty} b_n = x_0^+.$$

On the other hand, Example 5.18 i) tells

$$x_0^+ - x_0^- = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0,$$

so $x_0^- = x_0^+$. Let x_0 denote this number. Since f is continuous, and using the Substitution theorem (Theorem 9, p. 138), we have

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x_0).$$

But $f(a_n) < 0 < f(b_n)$, so the First comparison theorem (Theorem 4, p. 137) for $\{f(a_n)\}$ and $\{f(b_n)\}$ gives

$$\lim_{n\to\infty} f(a_n) \leq 0 \quad \text{and} \quad \lim_{n\to\infty} f(b_n) \geq 0.$$

As $0 \leq f(x_0) \leq 0$, we obtain $f(x_0) = 0$.
In conclusion, if f is strictly monotone on $[a, b]$ it must be injective by Proposition 2.8, which forces the zero to be unique. □

Some comments on this theorem might prove useful. We remark first that without the hypothesis of continuity on the closed interval $[a, b]$, the condition $f(a)f(b) < 0$ would not be enough to ensure the presence of a zero. The function $f: [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} -1 & \text{for } x = 0, \\ +1 & \text{for } 0 < x \leq 1 \end{cases}$$

takes values of discordant sign at the end-points but never vanishes; it has a jump point at $a = 0$.

Secondly, $f(a)f(b) < 0$ is a sufficient requirement only, and not a necessary one, to have a zero. The continuous map $f(x) = (2x - 1)^2$ vanishes on $[0, 1]$ despite being positive at both ends of the interval.

Thirdly, the halving procedure used in the proof can be transformed into an algorithm of approximation, known in Numerical Analysis under the name *Bisection method.*

A first application of the Theorem of existence of zeroes comes next.

Example 4.24

The function $f(x) = x^4 + x^3 - 1$ on $[0,1]$ is a polynomial, hence continuous. As $f(0) = -1$ and $f(1) = 1$, f must vanish somewhere on $[0,1]$. The zero is unique because the map is strictly increasing (it is sum of the strictly increasing functions $y = x^4$ and $y = x^3$, and of the constant function $y = -1$). □

Our theorem can be generalised usefully as follows.

Corollary 4.25 *Let f be continuous on the interval I and suppose it admits non-zero limits (finite or infinite) that are different in sign for x tending to the end-points of I. Then f has a zero in I, which is unique if f is strictly monotone on I.*

Proof. The result is a consequence of Theorems 4.2 and 4.23 (Existence of zeroes). For more details see Appendix A.3.2, p. 444. □

Example 4.26

Consider the map $f(x) = x + \log x$, defined on $I = (0, +\infty)$. The functions $y = x$ and $y = \log x$ are continuous and strictly increasing on I, and so is f. Since $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$, f has exactly one zero on its domain. □

Corollary 4.27 *Consider f and g continuous maps on the closed bounded interval $[a,b]$. If $f(a) < g(a)$ and $f(b) > g(b)$, there exists at least one point x_0 in the open interval (a,b) with*

$$f(x_0) = g(x_0). \tag{4.14}$$

Proof. Consider the auxiliary function $h(x) = f(x) - g(x)$, which is continuous in $[a,b]$ as sum of continuous maps. By assumption, $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$. So, h satisfies the Theorem of existence of zeroes and admits in (a,b) a point x_0 such that $h(x_0) = 0$. But this is precisely (4.14).

Note that if h is strictly increasing on $[a,b]$, the solution of (4.14) has to be unique in the interval. □

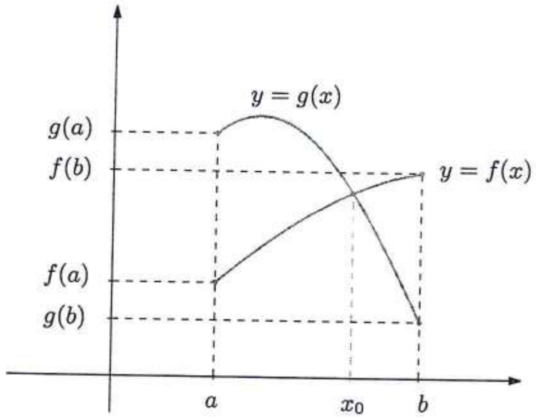

Figure 4.6. Illustration of Corollary 4.27

Example 4.28

Solve the equation

$$\cos x = x. \tag{4.15}$$

For any real x, $-1 \leq \cos x \leq 1$, so the equation cannot be solved when $x < -1$ or $x > 1$. Similarly, no solution exists on $[-1, 0)$, because $\cos x$ is positive while x is negative on that interval. Therefore the solutions, if any, must hide in $[0, 1]$: there the functions $f(x) = x$ and $g(x) = \cos x$ are continuous and $f(0) = 0 < 1 = g(0)$, $f(1) = 1 > \cos 1 = g(1)$ (cosine is 1 only for multiples of 2π). The above corollary implies that equation (4.15) has a solution in $(0, 1)$. There can be no other solution, for f is strictly increasing and g strictly decreasing on $[0, 1]$, making $h(x) = f(x) - g(x)$ strictly increasing. □

When one of the functions is a constant, the corollary implies this result.

Theorem 4.29 (Intermediate value theorem) *If a function f is continuous on the closed and bounded interval $[a, b]$, it assumes all values between $f(a)$ and $f(b)$.*

Proof. When $f(a) = f(b)$ the statement is trivial, so assume first $f(a) < f(b)$. Call z an arbitrary value between $f(a)$ and $f(b)$ and define the constant map $g(x) = z$. From $f(a) < z < f(b)$ we have $f(a) < g(a)$ and $f(b) > g(b)$. Corollary 4.27, applied to f and g in the interval $[a, b]$, yields a point x_0 in $[a, b]$ such that $f(x_0) = g(x_0) = z$.
If $f(a) > f(b)$, we just swap the roles of f and g. □

The Intermediate value theorem has, among its consequences, the remarkable fact that a continuous function maps intervals to intervals. This is the content of the next result.

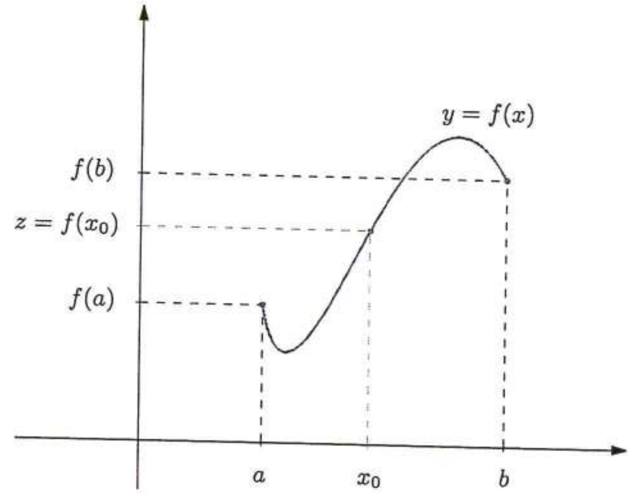

Figure 4.7. Intermediate value theorem

Corollary 4.30 *Let f be continuous on an interval I. The range $f(I)$ of I under f is an interval delimited by $\inf_I f$ and $\sup_I f$.*

Proof. A subset of $\mathbb{R}$ is an interval if and only if it contains the interval $[\alpha, \beta]$ as subset, for any $\alpha < \beta$.
Let then $y_1 < y_2$ be points of $f(I)$. There exist in I two (necessarily distinct) pre-images x_1 and x_2, i.e., $f(x_1) = y_1$, $f(x_2) = y_2$. If $J \subseteq I$ denotes the closed interval between x_1 and x_2, we need only to apply the Intermediate value theorem to f restricted to J, which yields $[y_1, y_2] \subseteq f(J) \subseteq f(I)$. The range $f(I)$ is then an interval, and according to Definition 2.3 its end-points are $\inf_I f$ and $\sup_I f$. □

Either one of $\inf_I f$, $\sup_I f$ may be finite or infinite, and may or not be an element of the interval itself. If, say, $\inf_I f$ belongs to the range, the function admits minimum on I (and the same for $\sup_I f$).

In case I is open or half-open, its image $f(I)$ can be an interval of any kind. Let us see some examples. Regarding $f(x) = \sin x$ on the open bounded $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, the image $f(I) = (-1, 1)$ is open and bounded. Yet under the same map, the image of the open bounded set $(0, 2\pi)$ is $[-1, 1]$, bounded but closed. Take now $f(x) = \tan x$: it maps the bounded interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ to the unbounded one $(-\infty, +\infty)$. Simple examples can be built also for unbounded I.

But if I is a closed bounded interval, its image under a continuous map cannot be anything but a closed bounded interval. More precisely, the following fundamental result holds, whose proof is given in Appendix A.3.2, p. 443.

Theorem 4.31 (Weierstrass) *A continuous map f on a closed and bounded interval $[a,b]$ is bounded and admits minimum and maximum*

$$m = \min_{x\in[a,b]} f(x) \qquad \text{and} \qquad M = \max_{x\in[a,b]} f(x).$$

Consequently,

$$f([a,b]) = [m,M]. \tag{4.16}$$

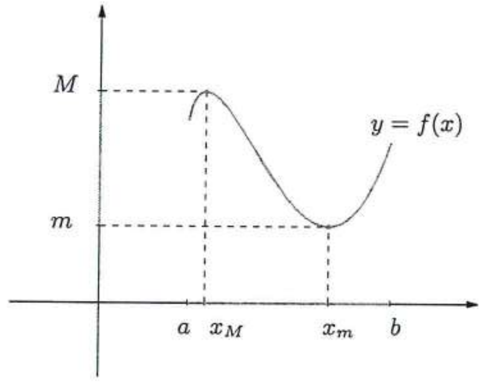

Figure 4.8. The Theorem of Weierstrass

In conclusion to this section, we present two results about invertibility (their proofs may be found in Appendix A.3.2, p. 445). We saw in Sect. 2.4 that a strictly monotone function is also one-to-one (invertible), and in general the opposite implication does not hold. Nevertheless, when speaking of continuous functions the notions of strict monotonicity and injectivity coincide. Moreover, the inverse function is continuous on its domain of definition.

Theorem 4.32 *A continuous function f on an interval I is one-to-one if and only if it is strictly monotone.*

Theorem 4.33 *Let f be continuous and invertible on an interval I. Then the inverse f^{-1} is continuous on the interval $J = f(I)$.*

Theorem 4.33 guarantees, by the way, the continuity of the inverse trigonometric functions $y = \arcsin x$, $y = \arccos x$ and $y = \arctan x$ on their domains, and of the logarithm $y = \log_a x$ on $\mathbb{R}_+$ as well, as inverse of the exponential $y = a^x$. These facts were actually already known from Proposition 3.20.

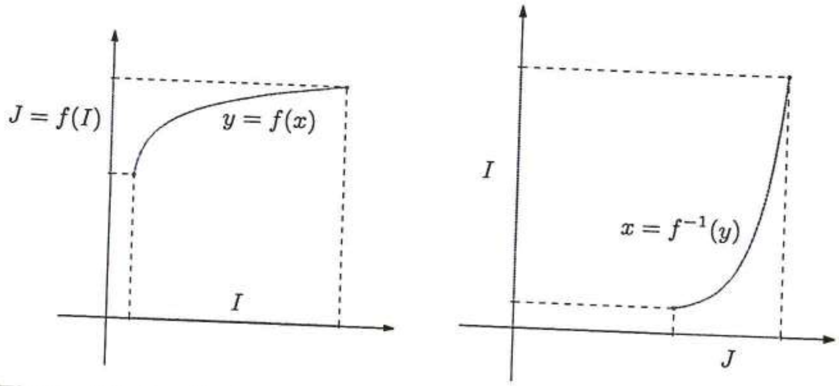

Figure 4.9. Graph of a continuous invertible map (left) and its inverse (right)

4.4 Exercises

1. *Compute the following limits using the Comparison theorems:*

a) $\displaystyle\lim_{x\to+\infty} \frac{\cos x}{\sqrt{x}}$

b) $\displaystyle\lim_{x\to+\infty} \left(\sqrt{x} + \sin x\right)$

c) $\displaystyle\lim_{x\to-\infty} \frac{2x - \sin x}{3x + \cos x}$

d) $\displaystyle\lim_{x\to+\infty} \frac{[x]}{x}$

e) $\displaystyle\lim_{x\to0} \sin x \cdot \sin \frac{1}{x}$

f) $\displaystyle\lim_{x\to0} \frac{x - \tan x}{x^2}$

2. *Determine the limits:*

a) $\displaystyle\lim_{x\to0} \frac{x^4 - 2x^3 + 5x}{x^5 - x}$

b) $\displaystyle\lim_{x\to+\infty} \frac{x + 3}{x^3 - 2x + 5}$

c) $\displaystyle\lim_{x\to-\infty} \frac{x^3 + x^2 + x}{2x^2 - x + 3}$

d) $\displaystyle\lim_{x\to+\infty} \frac{2x^2 + 5x - 7}{5x^2 - 2x + 3}$

e) $\displaystyle\lim_{x\to-1} \frac{x + 1}{\sqrt{6x^2 + 3} + 3x}$

f) $\displaystyle\lim_{x\to2} \frac{\sqrt[3]{10 - x} - 2}{x - 2}$

g) $\displaystyle\lim_{x\to+\infty} \left(\sqrt{x + 1} - \sqrt{x}\right)$

h) $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x} + x}{x}$

i) $\displaystyle\lim_{x\to-\infty} \left(\sqrt[3]{x + 1} - \sqrt[3]{x - 1}\right)$

ℓ) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2 + 3}}{4x + 2}$

3. *Relying on the fundamental limits, compute:*

a) $\displaystyle\lim_{x\to0} \frac{\sin^2 x}{x}$

b) $\displaystyle\lim_{x\to0} \frac{x \tan x}{1 - \cos x}$

c) $\lim_{x\to 0} \frac{\sin 2x - \sin 3x}{4x}$

d) $\lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{2x^2}$

e) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

f) $\lim_{x\to 0} \frac{\cos(\tan x) - 1}{\tan x}$

g) $\lim_{x\to 1} \frac{\cos\frac{\pi}{2}x}{1-x}$

h) $\lim_{x\to \frac{\pi}{2}} \frac{\sin x - 1}{\left(\frac{\pi}{2}-x\right)^2}$

i) $\lim_{x\to \pi} \frac{\cos x + 1}{\cos 3x + 1}$

ℓ) $\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1-\tan x}}{\sin x}$

4. *Calculate:*

a) $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$

b) $\lim_{x\to 0} \frac{e^{2x}-1}{e^{3x}-1}$

c) $\lim_{x\to e} \frac{\log x - 1}{x - e}$

d) $\lim_{x\to +\infty} \frac{e^x}{e^x - 1}$

e) $\lim_{x\to 0^+} \frac{2e^{2x}-1}{2x}$

f) $\lim_{x\to 1} \frac{\log x}{e^x - e}$

g) $\lim_{x\to 0} \frac{\sqrt[5]{1+3x}-1}{x}$

h) $\lim_{x\to -1} \frac{x+1}{\sqrt[4]{x+17}-2}$

5. *Compute the limits:*

a) $\lim_{x\to +\infty} \frac{x^{5/2} - 2x\sqrt{x} + 1}{2\sqrt{x^5} - 1}$

b) $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$

c) $\lim_{x\to 0} \left(\operatorname{cotan} x - \frac{1}{\sin x}\right)$

d) $\lim_{x\to +\infty} \sqrt{x}\left(\sqrt{x+1} - \sqrt{x-1}\right)$

e) $\lim_{x\to +\infty} \left(\frac{x-1}{x+3}\right)^{x-2}$

f) $\lim_{x\to 0} (1+x)^{\operatorname{cotan} x}$

g) $\lim_{x\to 5} \frac{x-5}{\sqrt{x}-\sqrt{5}}$

h) $\lim_{x\to -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$

i) $\lim_{x\to 0} \left(\frac{1}{x\tan x} - \frac{1}{x\sin x}\right)$

ℓ) $\lim_{x\to +\infty} xe^x \sin\left(e^{-x}\sin\frac{2}{x}\right)$

m) $\lim_{x\to +\infty} x(2+\sin x)$

n) $\lim_{x\to -\infty} xe^{\sin x}$

6. *Determine the domain of the functions below and their limit behaviour at the end-points of the domain:*

a) $f(x) = \frac{x^3 - x^2 + 3}{x^2 + 3x + 2}$

b) $f(x) = \frac{e^x}{1+x^4}$

c) $f(x) = \log\left[1 + \exp\left(\frac{x^2+1}{x}\right)\right]$

d) $f(x) = \sqrt[3]{x}e^{-x^2}$

4.4.1 Solutions

1. *Limits:*

a) 0; b) $+\infty$.

c) We have

$$\lim_{x\to-\infty} \frac{2x - \sin x}{3x + \cos x} = \lim_{x\to-\infty} \frac{x\,(2 - \frac{\sin x}{x})}{x\,(3 + \frac{\cos x}{x})} = \frac{2}{3}$$

because $\lim\limits_{x\to-\infty} \dfrac{\sin x}{x} = \lim\limits_{x\to-\infty} \dfrac{\cos x}{x} = 0$ by Corollary 4.7.

d) From $[x] \le x < [x] + 1$ (Example 2.1 vii)) one deduces straightaway $x - 1 < [x] \le x$, whence

$$\frac{x-1}{x} < \frac{[x]}{x} \le 1$$

for $x > 0$. Therefore, the Second comparison theorem 4.5 gives

$$\lim_{x\to+\infty} \frac{[x]}{x} = 1.$$

e) 0.

f) First of all $f(x) = \dfrac{x - \tan x}{x^2}$ is an odd map, so $\lim\limits_{x\to0+} f(x) = -\lim\limits_{x\to0^-} f(x)$. Let now $0 < x < \frac{\pi}{2}$. From

$$\sin x < x < \tan x$$

(see Example 4.6 i) for a proof) it follows

$$\sin x - \tan x < x - \tan x < 0, \qquad \text{that is,} \qquad \frac{\sin x - \tan x}{x^2} < \frac{x - \tan x}{x^2} < 0.$$

Secondly,

$$\lim_{x\to0+} \frac{\sin x - \tan x}{x^2} = \lim_{x\to0+} \frac{\sin x\,(\cos x - 1)}{x^2 \cos x} = \lim_{x\to0+} \frac{\sin x}{\cos x}\,\frac{\cos x - 1}{x^2} = 0\,.$$

Thus the Second comparison theorem 4.5 makes us conclude that

$$\lim_{x\to0+} \frac{x - \tan x}{x^2} = 0,$$

therefore the required limit is 0.

2. *Limits:*

a) -5; b) 0.

c) Simple algebraic operations give

$$\lim_{x\to-\infty}\frac{x^3+x^2+x}{2x^2-x+3}=\lim_{x\to-\infty}\frac{x^3\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\left(2-\frac{1}{x}+\frac{3}{x^2}\right)}=\lim_{x\to-\infty}\frac{x}{2}=-\infty\,.$$

d) $\frac{2}{5}$.

e) Rationalising the denominator we see

$$\begin{aligned}\lim_{x\to-1}\frac{x+1}{\sqrt{6x^2+3}+3x}&=\lim_{x\to-1}\frac{(x+1)(\sqrt{6x^2+3}-3x)}{6x^2+3-9x^2}\\&=\lim_{x\to-1}\frac{(x+1)(\sqrt{6x^2+3}-3x)}{3(1-x)(1+x)}=1\,.\end{aligned}$$

f) Use the relation $a^3-b^3=(a-b)(a^2+ab+b^2)$ in

$$\begin{aligned}\lim_{x\to2}\frac{\sqrt[3]{10-x}-2}{x-2}&=\lim_{x\to2}\frac{10-x-8}{(x-2)(\sqrt[3]{(10-x)^2}+2\sqrt[3]{10-x}+4)}\\&=\lim_{x\to2}\frac{-1}{\sqrt[3]{(10-x)^2}+2\sqrt[3]{10-x}+4}=-\frac{1}{12}\,.\end{aligned}$$

g) 0; h) 1; i) 0.

ℓ) We have

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+3}}{4x+2}=\lim_{x\to-\infty}\frac{|x|\sqrt{2+\frac{3}{x^2}}}{x\left(4+\frac{2}{x}\right)}=\frac{\sqrt{2}}{4}\lim_{x\to-\infty}\frac{-x}{x}=-\frac{\sqrt{2}}{4}\,.$$

3. *Limits:*

a) 0; b) 2.

c) We manipulate the expression so to obtain a fundamental limit:

$$\lim_{x\to0}\frac{\sin2x-\sin3x}{4x}=\lim_{x\to0}\frac{\sin2x}{4x}-\lim_{x\to0}\frac{\sin3x}{4x}=\frac{1}{2}-\frac{3}{4}=-\frac{1}{4}\,.$$

d) We use the cosine's fundamental limit:

$$\lim_{x\to0^+}\frac{1-\cos\sqrt{x}}{2x^2}=\lim_{x\to0^+}\frac{1-\cos\sqrt{x}}{x}\lim_{x\to0^+}\frac{1}{2x}=\frac{1}{2}\lim_{x\to0^+}\frac{1}{2x}=+\infty\,.$$

e) $\frac{1}{2}$.

f) Putting $y=\tan x$ and substituting,

$$\lim_{x\to0}\frac{\cos(\tan x)-1}{\tan x}=\lim_{y\to0}\frac{\cos y-1}{y}=\lim_{y\to0}\frac{\cos y-1}{y^2}\cdot y=0\,.$$

g) Letting $y = 1 - x$ transforms the limit into:

$$\lim_{x\to 1} \frac{\cos\frac{\pi}{2}x}{1-x} = \lim_{y\to 0} \frac{\cos\frac{\pi}{2}(1-y)}{y} = \lim_{y\to 0} \frac{\sin\frac{\pi}{2}y}{y} = \frac{\pi}{2}\,.$$

h) $-\frac{1}{2}$; i) $\frac{1}{9}$.

ℓ) One has

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1-\tan x}}{\sin x} &= \lim_{x\to 0} \frac{1+\tan x - 1 + \tan x}{\sin x\left(\sqrt{1+\tan x} + \sqrt{1-\tan x}\right)} \\ &= \frac{1}{2}\lim_{x\to 0} \frac{2\tan x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = 1\,.\end{aligned}$$

4. *Limits:*

a) $\frac{1}{\log 3}$; b) $\frac{2}{3}$.

c) By defining $y = x - \mathrm{e}$ we recover a known fundamental limit:

$$\begin{aligned}\lim_{x\to \mathrm{e}} \frac{\log x - 1}{x - \mathrm{e}} &= \lim_{y\to 0} \frac{\log(y+\mathrm{e}) - 1}{y} = \lim_{y\to 0} \frac{\log \mathrm{e}\,(1 + y/\mathrm{e}) - 1}{y} \\ &= \lim_{y\to 0} \frac{\log(1+y/\mathrm{e})}{y} = \frac{1}{\mathrm{e}}\,.\end{aligned}$$

Another possibility is to set $z = x/\mathrm{e}$:

$$\lim_{x\to \mathrm{e}} \frac{\log x - 1}{x - \mathrm{e}} = \lim_{z\to 1} \frac{\log(\mathrm{e}z) - 1}{\mathrm{e}(z-1)} = \frac{1}{\mathrm{e}}\lim_{z\to 1} \frac{\log z}{z-1} = \frac{1}{\mathrm{e}}\,.$$

d) 1.

e) We have

$$\begin{aligned}\lim_{x\to 0+} \frac{2\mathrm{e}^{2x} - 1}{2x} &= \lim_{x\to 0+} \frac{2(\mathrm{e}^{2x} - 1) + 1}{2x} \\ &= \lim_{x\to 0+} 2\frac{\mathrm{e}^{2x} - 1}{2x} + \lim_{x\to 0+} \frac{1}{2x} = 2 + \lim_{x\to 0+} \frac{1}{2x} = +\infty\,.\end{aligned}$$

f) Substitute $y = x - 1$, so that

$$\begin{aligned}\lim_{x\to 1} \frac{\log x}{\mathrm{e}^x - \mathrm{e}} &= \lim_{x\to 1} \frac{\log x}{\mathrm{e}(\mathrm{e}^{x-1} - 1)} \\ &= \lim_{y\to 0} \frac{\log(1+y)}{\mathrm{e}(\mathrm{e}^y - 1)} = \frac{1}{\mathrm{e}}\lim_{y\to 0} \frac{\log(1+y)}{y}\cdot\frac{y}{\mathrm{e}^y - 1} = \frac{1}{\mathrm{e}}\,.\end{aligned}$$

g) $\frac{3}{5}$.

h) The new variable $y = x + 1$ allows to recognize (4.13), so

$$\lim_{x\to -1} \frac{x+1}{\sqrt[4]{x+17}-2} = \lim_{y\to 0} \frac{y}{\sqrt[4]{y+16}-2} = \lim_{y\to 0} \frac{y}{2\left(\sqrt[4]{1+\frac{y}{16}}-1\right)}$$
$$= \frac{16}{2} \lim_{y\to 0} \frac{y/16}{\sqrt[4]{1+\frac{y}{16}}-1} = 8\cdot 4 = 32\,.$$

5. *Limits:*

a) $\frac{1}{2}$.

b) We have

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x\to 0} \frac{e^{-x}\left(e^{2x}-1\right)}{\sin x} = \lim_{x\to 0} e^{-x}\cdot \frac{e^{2x}-1}{2x}\cdot 2\cdot \frac{x}{\sin x} = 2\,.$$

c) One has

$$\lim_{x\to 0}\left(\cot x - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x^2}\cdot \frac{x}{\sin x}\cdot x = 0\,.$$

d) 1.

e) Start with

$$\lim_{x\to +\infty}\left(\frac{x-1}{x+3}\right)^{x-2} = \exp\left(\lim_{x\to +\infty}(x-2)\log\frac{x-1}{x+3}\right)$$
$$= \exp\left(\lim_{x\to +\infty}(x-2)\log\left(1-\frac{4}{x+3}\right)\right) = e^L\,.$$

Now define $y = \dfrac{1}{x+3}$, and substitute $x = \dfrac{1}{y} - 3$ at the exponent:

$$L = \lim_{y\to 0^+}\left(\frac{1}{y}-5\right)\log(1-4y) = \lim_{y\to 0^+}\left(\frac{\log(1-4y)}{y} - 5\log(1-4y)\right) = -4\,.$$

The required limit equals e^{-4}.

f) e; g) $2\sqrt{5}$.

h) We have

$$\lim_{x\to -\infty}\frac{3^x - 3^{-x}}{3^x + 3^{-x}} = \lim_{x\to -\infty}\frac{3^{-x}\left(3^{2x}-1\right)}{3^{-x}\left(3^{2x}+1\right)} = -1\,.$$

i) $-\frac{1}{2}$.

ℓ) Start by multiplying numerator and denominator by the same function:

$$\lim_{x\to +\infty} x e^x e^{-x}\sin\frac{2}{x}\cdot\frac{\sin\left(e^{-x}\sin\frac{2}{x}\right)}{e^{-x}\sin\frac{2}{x}} = \lim_{x\to +\infty} x\sin\frac{2}{x}\cdot \lim_{x\to +\infty}\frac{\sin\left(e^{-x}\sin\frac{2}{x}\right)}{e^{-x}\sin\frac{2}{x}}$$
$$= L_1\cdot L_2\,.$$

Now put $y = \frac{1}{x}$ in the first factor to get

$$L_1 = \lim_{y \to 0^+} \frac{\sin 2y}{y} = 2\,;$$

next, let $t = e^{-x} \sin \frac{2}{x}$. Since $t \to 0$ for $x \to +\infty$, by Corollary 4.7, the second factor is

$$L_2 = \lim_{t \to 0} \frac{\sin t}{t} = 1\,,$$

and eventually the limit is 2.

m) The fact that $-1 \le \sin x \le 1$ implies $1 \le 2 + \sin x \le 3$, so $x \le x(2 + \sin x)$ when $x > 0$. Since $\lim\limits_{x \to +\infty} x = +\infty$, the Second comparison theorem 4.8 gives $+\infty$ for an answer.

n) $-\infty$.

6. *Domains and limits:*

a) $\operatorname{dom} f = \mathbb{R} \setminus \{-2, -1\}$,
$\lim\limits_{x \to -2^\pm} f(x) = \pm\infty$, $\lim\limits_{x \to -1^\pm} f(x) = \pm\infty$, $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$.

b) The function is defined on the entire $\mathbb{R}$ and

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^x}{x^4} \cdot \frac{x^4}{1 + x^4} = \lim_{x \to +\infty} \frac{e^x}{x^4} = +\infty\,,$$
$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} e^x \cdot \lim_{x \to -\infty} \frac{1}{1 + x^4} = 0\,.$$

c) This function makes sense when $x \neq 0$ (because $1 + \exp\left(\frac{x^2+1}{x}\right) > 0$ for any non-zero x). As for the limits:

$$\lim_{x \to -\infty} f(x) = \log \lim_{x \to -\infty} \left(1 + \exp\left(\frac{x^2 + 1}{x}\right)\right) = \log 1 = 0\,,$$
$$\lim_{x \to +\infty} f(x) = \log \lim_{x \to +\infty} \left(1 + \exp\left(\frac{x^2 + 1}{x}\right)\right) = +\infty\,,$$
$$\lim_{x \to 0^-} f(x) = \log \lim_{x \to 0^-} \left(1 + \exp\left(\frac{x^2 + 1}{x}\right)\right) = \log 1 = 0\,,$$
$$\lim_{x \to 0^+} f(x) = \log \lim_{x \to 0^+} \left(1 + \exp\left(\frac{x^2 + 1}{x}\right)\right) = +\infty\,.$$

d) $\operatorname{dom} f = \mathbb{R}$; $\lim\limits_{x \to \pm\infty} f(x) = 0$.