5

# Local comparison of functions. Numerical sequences and series

In the first part of this chapter we learn how to compare the behaviour of two functions in the neighbourhood of a point. To this aim, we introduce suitable symbols – known as Landau symbols – that make the description of the possible types of behaviour easier. Of particular importance is the comparison between functions tending to 0 or $\infty$.

In the second part, we revisit some results on limits which we discussed in general for functions, and adapt them to the case of sequences. We present specific techniques for the analysis of the limiting behaviour of sequences. At last, numerical series are introduced and the main tools for the study of their convergence are provided.

## 5.1 Landau symbols

As customary by now, we denote by $c$ one of the symbols $x_0$ (real number), $x_0^+$, $x_0^-$, or $+\infty$, $-\infty$. By 'neighbourhood of $c$' we intend a neighbourhood – previously defined – of one of these symbols.

Let $f$ and $g$ be two functions defined in a neighbourhood of $c$, with the possible exception of the point $c$ itself. Let also $g(x) \neq 0$ for $x \neq c$. Assume the limit

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \ell \tag{5.1}$$

exists, finite or not. We introduce the following definition.

**Definition 5.1** *If $\ell$ is finite, we say that $f$ is* **controlled by** *$g$ for $x$ tending to $c$, and we shall use the notation*

$$f = O(g), \qquad x \to c,$$

*read as '$f$ is* **big o** *of $g$ for $x$ tending to $c$'.*

*This property can be made more precise by distinguishing three cases:*

*a) If $\ell$ is finite and non-zero, we say that $f$ has* **the same order of magnitude as** *$g$ (or* **is of the same order of magnitude**) *for $x$ tending to $c$; if so, we write*

$$f \asymp g, \qquad x \to c.$$

*As sub-case we have:*

*b) If $\ell = 1$, we call $f$* **equivalent to** *$g$ for $x$ tending to $c$; in this case we use the notation*

$$f \sim g, \qquad x \to c.$$

*c) Eventually, if $\ell = 0$, we say that $f$ is* **negligible with respect to** *$g$ when $x$ goes to $c$; for this situation the symbol*

$$f = o(g), \qquad x \to c,$$

*will be used, spoken 'f is* **little o** *of g for x tending to c'.*

Not included in the previous definition is the case in which $\ell$ is infinite. But in such a case

$$\lim_{x \to c} \frac{g(x)}{f(x)} = \frac{1}{\ell} = 0,$$

so we can say that $g = o(f)$ for $x \to c$.

The symbols $O$, $\asymp$, $\sim$, $o$ are called **Landau symbols.**

**Remark 5.2** The Landau symbols can be defined under more general assumptions than those considered at present, i.e., the mere existence of the limit (5.1). For instance the expression $f = O(g)$ as $x \to c$ could be extended to mean that there is a constant $C > 0$ such that in a suitable neighbourhood $I$ of $c$

$$|f(x)| \le C|g(x)|, \qquad \forall x \in I, \ x \ne c.$$

The given definition is nevertheless sufficient for our purposes. □

**Examples 5.3**

i) Keeping in mind Examples 4.6, we have

$$\sin x \sim x, \ x \to 0, \qquad \text{in fact} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1,$$

$$\sin x = o(x), \ x \to +\infty, \qquad \text{since} \qquad \lim_{x \to +\infty} \frac{\sin x}{x} = 0;$$

ii) We have $\sin x = o(\tan x)$, $x \to \frac{\pi}{2}$ since

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\tan x} = \lim_{x \to \frac{\pi}{2}} \cos x = 0.$$

iii) One has $\cos x \asymp 2x - \pi$, $x \to \frac{\pi}{2}$, because

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{2x-\pi} = \lim_{t\to 0} \frac{\cos(t+\frac{\pi}{2})}{2t} = -\lim_{t\to 0} \frac{\sin t}{2t} = -\frac{1}{2}. \qquad \square$$

### Properties of the Landau symbols

i) It is clear from the definitions that the symbols $\asymp$, $\sim$, $o$ are particular instances of $O$, in the sense that

$$f \asymp g \Rightarrow f = O(g), \qquad f \sim g \Rightarrow f = O(g), \qquad f = o(g) \Rightarrow f = O(g)$$

for $x \to c$. Moreover the symbol $\sim$ is a subcase of $\asymp$

$$f \sim g \quad \Rightarrow \quad f \asymp g.$$

Observe that if $f \asymp g$, then (5.1) implies

$$\lim_{x\to c} \frac{f(x)}{\ell g(x)} = 1, \qquad \text{hence } f \sim \ell g.$$

ii) The following property is useful

$$\boxed{f \sim g \qquad \Longleftrightarrow \qquad f = g + o(g).} \tag{5.2}$$

By defining $h(x) = f(x) - g(x)$ in fact, so that $f(x) = g(x) + h(x)$, we have

$$\begin{aligned} f \sim g \quad &\Longleftrightarrow \quad \lim_{x\to c} \frac{f(x)}{g(x)} = 1 \quad \Longleftrightarrow \quad \lim_{x\to c} \left(\frac{f(x)}{g(x)} - 1\right) = 0 \\ &\Longleftrightarrow \quad \lim_{x\to c} \frac{h(x)}{g(x)} = 0 \quad \Longleftrightarrow \quad h = o(g). \end{aligned}$$

iii) Computations are simplified once we notice that for any constant $\lambda \neq 0$

$$\boxed{o(\lambda f) = o(f) \qquad \text{and} \qquad \lambda\, o(f) = o(f).} \tag{5.3}$$

In fact $g = o(\lambda f)$ means that $\lim_{x\to c} \frac{g(x)}{\lambda f(x)} = 0$, otherwise said $\lim_{x\to c} \frac{g(x)}{f(x)} = 0$, or $g = o(f)$. The remaining identity is proved in a similar way. Analogous properties to (5.3) hold for the symbol $O$.

Note that $o(f)$ and $O(f)$ do not indicate one specific function, rather a precise property of any map represented by one of the two symbols.

iv) Prescribing $f = o(1)$ amounts to asking that $f$ converge to 0 when $x \to c$. Namely

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{f(x)}{1} = 0.$$

Similarly $f = O(1)$ means $f$ converges to a finite limit for $x$ tending to $c$. More generally (compare Remark 5.2), $f = O(1)$ means that $f$ is *bounded* in a neighbourhood of $c$: that is to say, there exists a constant $C > 0$ such that

$$|f(x)| \le C, \qquad \forall x \in I,\ x \neq c,$$

$I$ being a suitable neighbourhood of $c$.

v) The continuity of a function $f$ at a point $x_0$ can be expressed by means of the symbol $o$ in the equivalent form

$$f(x) = f(x_0) + o(1), \quad x \to x_0. \tag{5.4}$$

Recalling (3.9) in fact, we have

$$\lim_{x\to x_0} f(x) = f(x_0) \quad \Longleftrightarrow \quad \lim_{x\to x_0} \big(f(x) - f(x_0)\big) = 0$$
$$\Longleftrightarrow \quad f(x) - f(x_0) = o(1), \quad x \to x_0.$$

**The algebra of "little $o$'s"**

i) Let us compare the behaviour of the monomials $x^n$ as $x \to 0$:

$$\boxed{x^n = o(x^m), \quad x \to 0, \qquad \Longleftrightarrow \qquad n > m.}$$

In fact

$$\lim_{x\to 0} \frac{x^n}{x^m} = \lim_{x\to 0} x^{n-m} = 0 \ \text{ if and only if } \ n - m > 0.$$

Therefore *when $x \to 0$, the bigger of two powers of $x$ is negligible.*

ii) Now consider the limit when $x \to \pm\infty$. Proceeding as before we obtain

$$\boxed{x^n = o(x^m), \quad x \to \pm\infty, \qquad \Longleftrightarrow \qquad n < m.}$$

So, *for $x \to \pm\infty$, the lesser power of $x$ is negligible.*

iii) The symbols of Landau allow to simplify algebraic formulas quite a lot when studying limits. Consider for example the limit for $x \to 0$. The following properties, which define a special *"algebra of little o's"*, hold. Their proof is left to the reader as an exercise:

$$\begin{array}{|lll|}\hline \text{a)} & o(x^n) \pm o(x^n) = o(x^n)\,; & \\ \text{b)} & o(x^n) \pm o(x^m) = o(x^p)\,, & \text{with } p = \min(n,m)\,; \\ \text{c)} & o(\lambda x^n) = o(x^n), & \text{for each } \lambda \in \mathbb{R}\setminus\{0\}\,; \\ \hline\end{array} \tag{5.5}$$

| | | |
|---|---|---|
| d) | $\varphi(x)o(x^n) = o(x^n)$ | if $\varphi$ is bounded in a neighbourhood of $x = 0$; |
| e) | $x^m o(x^n) = o(x^{m+n})$; | |
| f) | $o(x^m)o(x^n) = o(x^{m+n})$; | |
| g) | $[o(x^n)]^k = o(x^{kn})$. | |

### Fundamental limits

The fundamental limits in the Table of p. 106 can be reformulated using the symbols of Landau:

| | | | | |
|---|---|---|---|---|
| $\sin x \sim x,$ | $x \to 0;$ | | | |
| $1 - \cos x \asymp x^2,$ | $x \to 0;$ | precisely, | $1 - \cos x \sim \frac{1}{2}x^2,$ | $x \to 0;$ |
| $\log(1+x) \sim x,$ | $x \to 0;$ | equivalently, | $\log x \sim x - 1,$ | $x \to 1;$ |
| $e^x - 1 \sim x,$ | $x \to 0;$ | | | |
| $(1+x)^\alpha - 1 \sim \alpha x,$ | $x \to 0.$ | | | |

With (5.2), and taking property (5.5) c) into account, these relations read:

| | | | |
|---|---|---|---|
| $\sin x = x + o(x),$ | $x \to 0;$ | | |
| $1 - \cos x = \frac{1}{2}x^2 + o(x^2),$ | $x \to 0,$ | or $\cos x = 1 - \frac{1}{2}x^2 + o(x^2),$ | $x \to 0;$ |
| $\log(1+x) = x + o(x),$ | $x \to 0,$ | or $\log x = x - 1 + o(x-1),$ | $x \to 1;$ |
| $e^x = 1 + x + o(x),$ | $x \to 0;$ | | |
| $(1+x)^\alpha = 1 + \alpha x + o(x),$ | $x \to 0.$ | | |

Besides, we shall prove in Sect. 6.11 that:

| | | | |
|---|---|---|---|
| a) | $x^\alpha = o(e^x),$ | $x \to +\infty,$ | $\forall \alpha \in \mathbb{R};$ |
| b) | $e^x = o(\lvert x\rvert^\alpha),$ | $x \to -\infty,$ | $\forall \alpha \in \mathbb{R};$ |
| c) | $\log x = o(x^\alpha),$ | $x \to +\infty,$ | $\forall \alpha > 0;$ |
| d) | $\log x = o\left(\dfrac{1}{x^\alpha}\right),$ | $x \to 0^+,$ | $\forall \alpha > 0.$ |

(5.6)

### Examples 5.4

i) From $e^t = 1 + t + o(t)$, $t \to 0$, by setting $t = 5x$ we have $e^{5x} = 1 + 5x + o(5x)$, i.e., $e^{5x} = 1 + 5x + o(x)$, $x \to 0$. In other words $e^{5x} - 1 \sim 5x$, $x \to 0$.

ii) Setting $t = -3x^2$ in $(1+t)^{1/2} = 1 + \frac{1}{2}t + o(t)$, $t \to 0$, we obtain $(1-3x^2)^{1/2} = 1 - \frac{3}{2}x^2 + o(-3x^2) = 1 - \frac{3}{2}x^2 + o(x^2)$, $x \to 0$. Thus $(1-3x^2)^{1/2} - 1 \sim -\frac{3}{2}x^2$, $x \to 0$.

iii) The relation $\sin t = t + o(t)$, $t \to 0$, implies, by putting $t = 2x$, $x \sin 2x = x(2x + o(2x)) = 2x^2 + o(x^2)$, $x \to 0$. Then $x \sin 2x \sim 2x^2$, $x \to 0$. □

We explain now how to use the symbols of Landau for calculating limits. All maps dealt with below are supposed to be defined, and not to vanish, on a neighbourhood of $c$, except possibly at $c$.

**Proposition 5.5** *Let us consider the limits*

$$\lim_{x\to c} f(x)g(x) \qquad \textit{and} \qquad \lim_{x\to c} \frac{f(x)}{g(x)}.$$

*Given functions $\tilde{f}$ and $\tilde{g}$ such that $\tilde{f} \sim f$ and $\tilde{g} \sim g$ for $x \to c$, then*

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \tilde{f}(x)\tilde{g}(x), \tag{5.7}$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\tilde{f}(x)}{\tilde{g}(x)}. \tag{5.8}$$

Proof. Start with (5.7). Then

$$\begin{aligned}\lim_{x\to c} f(x)g(x) &= \lim_{x\to c} \frac{f(x)}{\tilde{f}(x)}\tilde{f}(x)\frac{g(x)}{\tilde{g}(x)}\tilde{g}(x)\\ &= \lim_{x\to c} \frac{f(x)}{\tilde{f}(x)} \lim_{x\to c} \frac{g(x)}{\tilde{g}(x)} \lim_{x\to c} \tilde{f}(x)\tilde{g}(x).\end{aligned}$$

From the definition of $\tilde{f} \sim f$ and $\tilde{g} \sim g$ the result follows. The proof of (5.8) is completely analogous. □

**Corollary 5.6** *Consider the limits*

$$\lim_{x\to c} \big(f(x) + f_1(x)\big)\big(g(x) + g_1(x)\big) \qquad \textit{and} \qquad \lim_{x\to c} \frac{f(x) + f_1(x)}{g(x) + g_1(x)}.$$

*If $f_1 = o(f)$ and $g_1 = o(g)$ when $x \to c$, then*

$$\lim_{x\to c} \big(f(x) + f_1(x)\big)\big(g(x) + g_1(x)\big) = \lim_{x\to c} f(x)g(x), \tag{5.9}$$

$$\lim_{x\to c} \frac{f(x) + f_1(x)}{g(x) + g_1(x)} = \lim_{x\to c} \frac{f(x)}{g(x)}. \tag{5.10}$$

Proof. Set $\tilde{f} = f + f_1$; by assumption $\tilde{f} = f + o(f)$, so from (5.2) one has $\tilde{f} \sim f$. Similarly, putting $\tilde{g} = g + g_1$ yields $\tilde{g} \sim g$. The claim follows from the previous Proposition. □

The meaning of these properties is clear: when computing the limit of a product, we may substitute each factor with an equivalent function. Alternatively, one may ignore negligible summands with respect to others within one factor. In a similar way one can handle the limit of a quotient, numerator and denominator now being the 'factors'.

**Examples 5.7**

i) Compute

$$\lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 3x}.$$

From the equivalence $1-\cos t \sim \frac{1}{2}t^2$, $t \to 0$, the substitution $t = 2x$ gives

$$1-\cos 2x \sim 2x^2, \quad x \to 0.$$

Putting $t = 3x$ in $\sin t \sim t$, $t \to 0$, we obtain $\sin 3x \sim 3x$, $x \to 0$, hence

$$\sin^2 3x \sim 9x^2, \quad x \to 0.$$

Therefore (5.8) implies

$$\lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 3x} = \lim_{x\to 0} \frac{2x^2}{9x^2} = \frac{2}{9}.$$

ii) Evaluate

$$\lim_{x\to 0} \frac{\sin 2x + x^3}{4x + 5\log(1+x^2)}.$$

We shall show that for $x \to 0$, $x^3$ is negligible with respect to $\sin 2x$, and similarly $5\log(1+x^2)$ is negligible with respect to $4x$. With that, we can use the previous corollary and conclude

$$\lim_{x\to 0} \frac{\sin 2x + x^3}{4x + 5\log(1+x^2)} = \lim_{x\to 0} \frac{\sin 2x}{4x} = \frac{1}{2}.$$

Recall $\sin 2x \sim 2x$ for $x \to 0$; thus

$$\lim_{x\to 0} \frac{x^3}{\sin 2x} = \lim_{x\to 0} \frac{x^3}{2x} = 0,$$

that is to say $x^3 = o(\sin 2x)$ for $x \to 0$. On the other hand, since $\log(1+t) \sim t$ for $t \to 0$, writing $t = x^2$ yields $\log(1+x^2) \sim x^2$ when $x \to 0$. Then

$$\lim_{x\to 0} \frac{5\log(1+x^2)}{4x} = \lim_{x\to 0} \frac{5x^2}{4x} = 0,$$

i.e., $5\log(1+x^2) = o(4x)$ for $x \to 0$. □

These 'simplification' rules hold only in the case of products and quotients. *They do not apply to limits of sums or differences of functions.* Otherwise put, the fact that $\tilde{f} \sim f$ and $\tilde{g} \sim g$ when $x \to c$, does not allow to conclude that

$$\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c}[\tilde{f}(x) \pm \tilde{g}(x)].$$

For example set $f(x) = \sqrt{x^2+2x}$ and $g(x) = \sqrt{x^2-1}$ and consider the limit

$$\lim_{x\to+\infty}\left(\sqrt{x^2+2x} - \sqrt{x^2-1}\right).$$

Rationalisation turns this limit into

$$\lim_{x\to+\infty}\frac{(x^2+2x)-(x^2-1)}{\sqrt{x^2+2x}+\sqrt{x^2-1}} = \lim_{x\to+\infty}\frac{2x+1}{x\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x^2}}\right)} = 1.$$

Had we substituted to $f(x)$ the function $\tilde{f}(x) = x$, equivalent to $f$ for $x \to +\infty$, we would have obtained a different limit, actually a *wrong* one. In fact,

$$\lim_{x\to+\infty}\left(x - \sqrt{x^2-1}\right) = \lim_{x\to+\infty}\frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \lim_{x\to+\infty}\frac{1}{x(1+\sqrt{1-\frac{1}{x^2}})} = 0.$$

The reason for the mismatch lies in the *cancellation* of the leading term $x^2$ appearing in the numerator after rationalisation, which renders the terms of lesser degree important for the limit, even though they are negligible with respect to $x^2$ for $x \to +\infty$.

## 5.2 Infinitesimal and infinite functions

**Definition 5.8** *Let $f$ be a function defined in a neighbourhood of $c$, except possibly at $c$. Then $f$ is said* **infinitesimal** (*or* **an infinitesimal**) *at $c$ if*

$$\lim_{x\to c} f(x) = 0,$$

*i.e., if $f = o(1)$ for $x \to c$. The function $f$ is said* **infinite** *at $c$ if*

$$\lim_{x\to c} f(x) = \infty.$$

Let us introduce the following terminology to compare two infinitesimal or infinite maps.

**Definition 5.9** *Let $f$, $g$ be two infinitesimals at $c$.*
*If $f \asymp g$ for $x \to c$, $f$ and $g$ are said* **infinitesimals of the same order**.
*If $f = o(g)$ for $x \to c$, $f$ is called* **infinitesimal of bigger order than $g$**.
*If $g = o(f)$ for $x \to c$, $f$ is called* **infinitesimal of smaller order than $g$**.
*If none of the above are satisfied, $f$ and $g$ are said* **non-comparable** *infinitesimals.*

**Definition 5.10** *Let $f$ and $g$ be two infinite maps at $c$.*
*If $f \asymp g$ for $x \to c$, $f$ and $g$ are said to be* **infinite of the same order**.
*If $f = o(g)$ for $x \to c$, $f$ is called* **infinite of smaller order than** $g$.
*If $g = o(f)$ for $x \to c$, $f$ is called* **infinite of bigger order than** $g$.
*If none of the above are satisfied, the infinite functions $f$ and $g$ are said* **non-comparable**.

**Examples 5.11**

Bearing in mind the fundamental limits seen above, it is immediate to verify the following facts:

i) $e^x - 1$ is an infinitesimal of the same order as $x$ at the origin.

ii) $\sin x^2$ is an infinitesimal of bigger order than $x$ at the origin.

iii) $\dfrac{\sin x}{(1-\cos x)^2}$ is infinite of bigger order than $\dfrac{1}{x}$ at the origin.

iv) For every $\alpha > 0$, $e^x$ is infinite of bigger order than $x^\alpha$ for $x \to +\infty$.

v) For every $\alpha > 0$, $\log x$ is infinite of smaller order than $\dfrac{1}{x^\alpha}$ for $x \to 0^+$.

vi) The functions $f(x) = x \sin \frac{1}{x}$ and $g(x) = x$ are infinitesimal for $x$ tending to 0 (for $f$ recall Corollary 4.7). But the quotient $\dfrac{f(x)}{g(x)} = \sin \frac{1}{x}$ does not admit limit for $x \to 0$, for in any neighbourhood of 0 it attains every value between $-1$ and 1 infinitely many times. Therefore none of the conditions $f \asymp g$, $f = o(g)$, $g = o(f)$ hold for $x \to 0$. The two functions $f$ and $g$ are thus not comparable. □

Using a non-rigorous yet colourful language, we shall express the fact that $f$ is infinitesimal (or infinite) of bigger order than $g$ by saying that $f$ tends to 0 (or $\infty$) *faster* than $g$. This suggests to measure the speed at which an infinitesimal (or infinite) map converges to its limit value.

For that purpose, let us fix an infinitesimal (or infinite) map $\varphi$ defined in a neighbourhood of $c$ and particularly easy to compute. We shall use it as term of comparison ('test function') and in fact call it an **infinitesimal test function** (or **infinite test function**) at $c$. When the limit behaviour is clear, we refer to $\varphi$ as test function for brevity. The most common test functions (certainly *not* the only ones) are the following. If $c = x_0 \in \mathbb{R}$, we choose

$$\varphi(x) = x - x_0 \qquad \text{or} \qquad \varphi(x) = |x - x_0|$$

as infinitesimal test functions (the latter in case we need to consider non-integer powers of $\varphi$, see later), and

$$\varphi(x) = \frac{1}{x - x_0} \qquad \text{or} \qquad \varphi(x) = \frac{1}{|x - x_0|}$$

as infinite test functions. For $c = x_0^+$ $(c = x_0^-)$, we will choose as infinitesimal test function

$$\varphi(x) = x - x_0 \qquad (\varphi(x) = x_0 - x)$$

and as infinite test function

$$\varphi(x) = \frac{1}{x - x_0} \qquad (\varphi(x) = \frac{1}{x_0 - x}).$$

For $c = +\infty$, the infinitesimal and infinite test functions will respectively be

$$\varphi(x) = \frac{1}{x} \qquad \text{and} \qquad \varphi(x) = x,$$

while for $c = -\infty$, we shall take

$$\varphi(x) = \frac{1}{|x|} \qquad \text{and} \qquad \varphi(x) = |x|.$$

The definition of 'speed of convergence' of an infinitesimal or infinite $f$ depends on how $f$ compares to the powers of the infinitesimal or infinite test function. To be precise, we have the following definition

**Definition 5.12** *Let $f$ be infinitesimal (or infinite) at $c$. If there exists a real number $\alpha > 0$ such that*

$$f \asymp \varphi^\alpha, \quad x \to c, \tag{5.11}$$

*the constant $\alpha$ is called the* **order of $f$ at $c$ with respect to the infinitesimal (infinite) test function $\varphi$.**

Notice that if condition (5.11) holds, it determines the order uniquely. In the first case in fact, it is immediate to see that for any $\beta < \alpha$ one has $f = o(\varphi^\beta)$, while $\beta > \alpha$ implies $\varphi^\beta = o(f)$. A similar argument holds for infinite maps.

If $f$ has order $\alpha$ at $c$ with respect to the test function $\varphi$, then there is a real number $\ell \neq 0$ such that

$$\lim_{x \to c} \frac{f(x)}{\varphi^\alpha(x)} = \ell.$$

Rephrasing:

$$f \sim \ell\varphi^\alpha, \quad x \to c,$$

which is to say – recalling (5.2) – $f = \ell\varphi^\alpha + o(\ell\varphi^\alpha)$, for $x \to c$. For the sake of simplicity we can omit the constant $\ell$ in the symbol $o$, because if a function $h$ satisfies $h = o(\ell\varphi^\alpha)$, then $h = o(\varphi^\alpha)$ as well. Therefore

$$f = \ell\varphi^\alpha + o(\varphi^\alpha), \quad x \to c.$$

**Definition 5.13** *The function*

$$p(x) = \ell\varphi^\alpha(x) \tag{5.12}$$

*is called the* **principal part of the infinitesimal (infinite) map** $f$ **at** $c$ **with respect to the infinitesimal (infinite) test function** $\varphi$.

From the qualitative point of view the behaviour of the function $f$ in a small enough neighbourhood of $c$ coincides with the behaviour of its principal part (in geometrical terms, the two graphs resemble each other). With a suitable choice of test function $\varphi$, like one of those mentioned above, the behaviour of the function $\ell\varphi^\alpha(x)$ becomes immediately clear. So if one is able to determine the principal part of a function, even a complicated one, at a given point $c$, the local behaviour around that point is easily described.

We wish to stress that to find the order and the principal part of a function $f$ at $c$, one must start from the limit

$$\lim_{x\to c} \frac{f(x)}{\varphi^\alpha(x)}$$

and understand if there is a number $\alpha$ for which such limit – say $\ell$ – is finite and different from zero. If so, $\alpha$ is the required order, and the principal part of $f$ is given by (5.12).

**Examples 5.14**

i) The function $f(x) = \sin x - \tan x$ is infinitesimal for $x \to 0$. Using the basic equivalences of p. 127 and Proposition 5.5, we can write

$$\sin x - \tan x = \frac{\sin x\,(\cos x - 1)}{\cos x} \sim \frac{x \cdot \left(-\frac{1}{2}x^2\right)}{1} = -\frac{1}{2}x^3, \qquad x \to 0.$$

It follows that $f(x)$ is infinitesimal of order 3 at the origin with respect to the test function $\varphi(x) = x$; its principal part is $p(x) = -\frac{1}{2}x^3$.

ii) The function

$$f(x) = \sqrt{x^2+3} - \sqrt{x^2-1}$$

is infinitesimal for $x \to +\infty$. Rationalising the expression we get

$$f(x) = \frac{(x^2+3)-(x^2-1)}{\sqrt{x^2+3}+\sqrt{x^2-1}} = \frac{4}{x\left(\sqrt{1+\frac{3}{x^2}}+\sqrt{1-\frac{1}{x^2}}\right)}.$$

The right-hand side shows that if one chooses $\varphi(x) = \frac{1}{x}$ then

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi(x)} = 2.$$

Therefore $f$ is infinitesimal of first order for $x \to +\infty$ with respect to the test function $\frac{1}{x}$, with principal part $p(x) = \frac{2}{x}$.

iii) The function

$$f(x) = \sqrt{9x^5 + 7x^3 - 1}$$

is infinite when $x \to +\infty$. To determine its order with respect to $\varphi(x) = x$, we consider the limit

$$\lim_{x\to+\infty} \frac{f(x)}{x^\alpha} = \lim_{x\to+\infty} \frac{x^{\frac{5}{2}}\sqrt{9 + \frac{7}{x^2} - \frac{1}{x^5}}}{x^\alpha}.$$

By choosing $\alpha = \frac{5}{2}$ the limit becomes 3. So $f$ has order $\frac{5}{2}$ for $x \to +\infty$ with respect to the test function $\varphi(x) = x$. The principal part is $p(x) = 3x^{5/2}$. □

**Remark 5.15** The previous are typical instances of how to determine the order of a function with respect to some test map. The reader should not be mislead to believe that this is always possible. Given an infinitesimal or an infinite $f$ at $c$, and having chosen a corresponding test map $\varphi$, it may well happen that there is no real number $\alpha > 0$ satisfying $f \asymp \varphi^\alpha$ for $x \to c$. In such a case it is convenient to make a different choice of test function, one more suitable to describe the behaviour of $f$ around $c$. We shall clarify this fact with two examples.

Start by taking the function $f(x) = e^{2x}$ for $x \to +\infty$. Using (5.6) a), it follows immediately that $x^\alpha = o(e^{2x})$, whichever $\alpha > 0$ is considered. So it is not possible to determine an order for $f$ with respect to $\varphi(x) = x$: the exponential map grows too quickly for any polynomial function to keep up with it. But if we take as test function $\varphi(x) = e^x$ then clearly $f$ has order 2 with respect to $\varphi$.

Consider now $f(x) = x \log x$ for $x \to 0^+$. In (5.6) d) we claimed that

$$\lim_{x\to0^+} \frac{\log x}{\frac{1}{x^\beta}} = 0, \qquad \forall \beta > 0.$$

So in particular $f(x) = \dfrac{\log x}{1/x}$ is infinitesimal when $x \to 0^+$. Using the test function $\varphi(x) = x$ one sees that

$$\lim_{x\to0^+} \frac{x\log x}{x^\alpha} = \lim_{x\to0^+} \frac{\log x}{x^{\alpha-1}} = \begin{cases} 0 & \text{if } \alpha < 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Definition 5.9 yields that $f$ is an infinitesimal of *bigger* order than any power of $x$ with exponent less than one. At the same time it has *smaller* order than $x$ and all powers with exponent greater than one. In this case too, it is not possible to determine the order of $f$ with respect to $x$. The function $|f(x)| = x|\log x|$ goes to zero more slowly than $x$, yet faster than $x^\alpha$ for any $\alpha < 1$. Thus it can be used as alternative infinitesimal test map when $x \to 0^+$. □

## 5.3 Asymptotes

We now consider a function $f$ defined in a neighbourhood of $+\infty$ and wish to study its behaviour for $x \to +\infty$. A remarkable case is that in which $f$ behaves as a polynomial of first degree. Geometrically speaking, this corresponds to the fact that the graph of $f$ will more and more look like a straight line. Precisely, we suppose there exist two real numbers $m$ and $q$ such that

$$\boxed{\lim_{x\to+\infty} \big(f(x) - (mx+q)\big) = 0,} \tag{5.13}$$

or, using the symbols of Landau,

$$f(x) = mx + q + o(1), \qquad x \to +\infty.$$

We then say that the line $g(x) = mx + q$ is a **right asymptote** of the function $f$. The asymptote is called **oblique** if $m \neq 0$, **horizontal** if $m = 0$. In geometrical terms condition (5.13) tells that the vertical distance $d(x) = |f(x) - g(x)|$ between the graph of $f$ and the asymptote tends to 0 as $x \to +\infty$ (Fig. 5.1).

The asymptote's coefficients can be recovered using limits:

$$\boxed{m = \lim_{x\to+\infty} \frac{f(x)}{x} \qquad \text{and} \qquad q = \lim_{x\to+\infty} \big(f(x) - mx\big).} \tag{5.14}$$

The first relation comes from (5.13) noting that

$$0 = \lim_{x\to+\infty} \frac{f(x) - mx - q}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} - \lim_{x\to+\infty} \frac{mx}{x} - \lim_{x\to+\infty} \frac{q}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} - m,$$

while the second one follows directly from (5.13). The conditions (5.14) furnish the means to find the possible asymptote of a function $f$. If in fact both limits exist

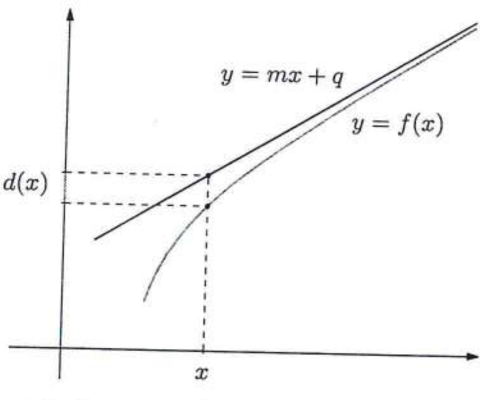


**Figure 5.1.** Graph of a function with its right asymptote

and are finite, $f$ admits $y = mx + q$ as a right asymptote. If only one of (5.14) is not finite instead, then $f$ will not have an asymptote.

Notice that if $f$ has an oblique asymptote, i.e., if $m \neq 0$, the first of (5.14) tells us that $f$ is infinite of order 1 with respect to the test function $\varphi(x) = x$ for $x \to +\infty$. The reader should beware that not all functions satisfying the latter condition do admit an oblique asymptote: the function $f(x) = x + \sqrt{x}$ for example is equivalent to $x$ for $x \to +\infty$, but has no asymptote since the second limit in (5.14) is $+\infty$.

**Remark 5.16** The definition of (linear) asymptote given above is a particular instance of the following. The function $f$ is called **asymptotic** to a function $g$ for $x \to +\infty$ if

$$\lim_{x\to+\infty} (f(x) - g(x)) = 0.$$

If (5.13) holds one can then say that $f$ is asymptotic to the line $g(x) = mx + q$. The function $f(x) = x^2 + \frac{1}{x}$ instead has no line as asymptote for $x \to +\infty$, but is nevertherless asymptotic to the parabola $g(x) = x^2$. □

In a similar fashion one defines oblique or horizontal asymptotes for $x \to -\infty$ (that is oblique or horizontal left asymptotes).

If the line $y = mx + q$ is an oblique or horizontal asymptote both for $x \to +\infty$ and $x \to -\infty$, we shall say that it is a **complete oblique** or **complete horizontal asymptote** for $f$.

Eventually, if at a point $x_0 \in \mathbb{R}$ one has $\lim_{x\to x_0} f(x) = \infty$, the line $x = x_0$ is called a **vertical asymptote** for $f$ at $x_0$. The distance between points on the graph of $f$ and on a vertical asymptote with the same $y$-coordinate converges to zero for $x \to x_0$. If the limit condition holds only for $x \to x_0^+$ or $x \to x_0^-$ we talk about a vertical **right** or **left** asymptote respectively.

**Examples 5.17**

i) Let $f(x) = \dfrac{x}{x+1}$. As

$$\lim_{x\to\pm\infty} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to-1^\pm} f(x) = \mp\infty,$$

the function has a horizontal asymptote $y = 1$ and a vertical asymptote $x = -1$.

ii) The map $f(x) = \sqrt{1+x^2}$ satisfies

$$\lim_{x\to\pm\infty} f(x) = +\infty, \qquad \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{|x|\sqrt{1+x^{-2}}}{x} = \pm 1$$

and

$$\lim_{x\to+\infty} \left(\sqrt{1+x^2} - x\right) = \lim_{x\to+\infty} \frac{1+x^2-x^2}{\sqrt{1+x^2}+x} = 0\,,$$

$$\lim_{x\to-\infty} \left(\sqrt{1+x^2} + x\right) = \lim_{x\to-\infty} \frac{1+x^2-x^2}{\sqrt{1+x^2}-x} = 0\,.$$

Therefore $f$ has an oblique asymptote for $x \to +\infty$ given by $y = x$, plus another one of equation $y = -x$ for $x \to -\infty$.

iii) Let $f(x) = x + \log x$. Since

$$\lim_{x\to 0^+} (x + \log x) = -\infty, \qquad \lim_{x\to +\infty} (x + \log x) = +\infty,$$

$$\lim_{x\to +\infty} \frac{x + \log x}{x} = 1, \qquad \lim_{x\to +\infty} (x + \log x - x) = +\infty,$$

the function has a vertical right asymptote $x = 0$ but no horizontal nor oblique asymptotes.

□

## 5.4 Further properties of sequences

We return to the study of the limit behaviour of sequences begun in Sect. 3.2. General theorems concerning functions apply to sequences as well (the latter being particular functions defined over the integers, after all). For the sake of completeness those results will be recalled, and adapted to the case of concern. We shall also state and prove other specific properties of sequences.

We say that a sequence $\{a_n\}_{n\geq n_0}$ satisfies a given property **eventually**, if there exists an integer $N \geq n_0$ such that the sequence $\{a_n\}_{n\geq N}$ satisfies that property. This definition allows for a more flexible study of sequences.

### Theorems on sequences

1. *Uniqueness of the limit*: the limit of a sequence, when defined, is unique.
2. *Boundedness*: a converging sequence is bounded.
3. *Existence of limit for monotone sequences*: if an eventually monotone sequence is bounded, then it converges; if not bounded then it diverges (to $+\infty$ if increasing, to $-\infty$ if decreasing).
4. *First comparison theorem*: let $\{a_n\}$ and $\{b_n\}$ be sequences with finite or infinite limits $\lim_{n\to\infty} a_n = \ell$ and $\lim_{n\to\infty} b_n = m$. If $a_n \leq b_n$ eventually, then $\ell \leq m$.
5. *Second comparison theorem ("Squeeze rule")*: let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \ell$. If $a_n \leq b_n \leq c_n$ eventually, then $\lim_{n\to\infty} b_n = \ell$.
6. *Theorem*: a sequence $\{a_n\}$ is infinitesimal, that is $\lim_{n\to\infty} a_n = 0$, if and only if the sequence $\{|a_n|\}$ is infinitesimal.
7. *Theorem*: let $\{a_n\}$ be an infinitesimal sequence and $\{b_n\}$ a bounded one. Then the sequence $\{a_n b_n\}$ is infinitesimal.

8. *Algebra of limits*: let $\{a_n\}$ and $\{b_n\}$ be such that $\lim_{n\to\infty} a_n = \ell$ and $\lim_{n\to\infty} b_n = m$ ($\ell$, $m$ finite or infinite). Then

$$\lim_{n\to\infty} (a_n \pm b_n) = \ell \pm m\,,$$
$$\lim_{n\to\infty} a_n\, b_n = \ell\, m\,,$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\ell}{m}, \quad \text{if } b_n \neq 0 \text{ eventually}\,,$$

each time the right-hand sides are defined according to the Table on p. 96.

9. *Substitution theorem*: let $\{a_n\}$ be a sequence with $\lim_{n\to\infty} a_n = \ell$ and suppose $g$ is a function defined in a neighbourhood of $\ell$:
   a) if $\ell \in \mathbb{R}$ and $g$ is continuous at $\ell$, then $\lim_{n\to\infty} g(a_n) = g(\ell)$;
   b) if $\ell \notin \mathbb{R}$ and $\lim_{x\to\ell} g(x) = m$ exists, then $\lim_{n\to\infty} g(a_n) = m$.

Proof. We shall only prove Theorem 2 since the others are derived adapting the similar proofs given for functions.
Let the sequence $\{a_n\}_{n\geq n_0}$ be given, and suppose it converges to $\ell \in \mathbb{R}$. With $\varepsilon = 1$ fixed, there exists an integer $n_1 \geq n_0$ so that $|a_n - \ell| < 1$ for all $n > n_1$. For such $n$'s then the triangle inequality (1.1) yields

$$|a_n| = |a_n - \ell + \ell| \leq |a_n - \ell| + |\ell| < 1 + |\ell|.$$

By putting $M = \max\{|a_{n_0}|, \ldots, |a_{n_1}|, 1 + |\ell|\}$ one obtains $|a_n| \leq M$, $\forall n \geq n_0$.
□

**Examples 5.18**

i) Consider the sequence $a_n = q^n$,where $q$ is a fixed number in $\mathbb{R}$. It goes under the name of *geometric sequence*. We claim that

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{if } |q| < 1, \\ 1 & \text{if } q = 1, \\ +\infty & \text{if } q > 1, \\ \text{does not exist} & \text{if } q \leq -1. \end{cases}$$

If either $q = 0$ or $q = 1$, the sequence is constant and thus trivially convergent to 0 or 1 respectively. When $q = -1$ the sequence is indeterminate.
Let $q > 1$: the sequence is now strictly increasing and so admits a limit. In order to show that the limit is indeed $+\infty$ we write $q = 1 + r$ with $r > 0$ and apply the binomial formula (1.13):

$$q^n = (1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k = 1 + nr + \sum_{k=2}^{n} \binom{n}{k} r^k.$$

As all terms in the last summation are positive, we obtain

$$(1+r)^n \geq 1 + nr\,, \qquad \forall n \geq 0\,, \tag{5.15}$$

called **Bernoulli inequality**[1]. Therefore $q^n \geq 1 + nr$; passing to the limit for $n \to \infty$ and using the First comparison theorem we can conclude.

Let us examine the case $|q| < 1$ with $q \neq 0$. We just saw that $\dfrac{1}{|q|} > 1$ implies $\lim\limits_{n\to\infty} \left(\dfrac{1}{|q|}\right)^n = +\infty$. The sequence $\{|q|^n\}$ is thus infinitesimal, and so is $\{q^n\}$.

At last, take $q < -1$. Since

$$\lim_{k\to\infty} q^{2k} = \lim_{k\to\infty} (q^2)^k = +\infty, \qquad \lim_{k\to\infty} q^{2k+1} = q \lim_{k\to\infty} q^{2k} = -\infty,$$

the sequence $q^n$ is indeterminate.

ii) Let $p$ be a fixed positive number and consider the sequence $\sqrt[n]{p}$. Applying the Substitution theorem with $g(x) = p^x$ we have

$$\boxed{\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} p^{1/n} = p^0 = 1\,.}$$

iii) Consider the sequence $\sqrt[n]{n}$; using once again the Substitution theorem together with (5.6) c), it follows that

$$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \exp\frac{\log n}{n} = \mathrm{e}^0 = 1.}$$

□

There are easy criteria to decide whether a sequence is infinitesimal or infinite. Among them, the following is the most widely employed.

**Theorem 5.19 (Ratio test)** *Let $\{a_n\}$ be a sequence for which $a_n > 0$ eventually. Suppose the limit*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$$

*exists, finite or infinite. If $q < 1$ then $\lim\limits_{n\to\infty} a_n = 0$; if $q > 1$ then $\lim\limits_{n\to\infty} a_n = +\infty$.*

[1] By the Principle of Induction, one can prove that (5.15) actually holds for any $r \geq -1$; see Appendix A.1, p. 427.

*Proof.* Suppose $a_n > 0$, $\forall n \geq n_0$. Take $q < 1$ and set $\varepsilon = 1 - q$. By definition of limit there exists an integer $n_\varepsilon \geq n_0$ such that for all $n > n_\varepsilon$

$$\frac{a_{n+1}}{a_n} < q + \varepsilon = 1\,, \quad \text{i.e.,} \quad a_{n+1} < a_n.$$

So the sequence $\{a_n\}$ is monotone decreasing eventually, and as such it admits a finite non-negative limit $\ell$. Now if $\ell$ were different from zero, the fact that

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\ell}{\ell} = 1$$

would contradict the assumption $q < 1$.
If $q > 1$, it is enough to consider the sequence $\{1/a_n\}$. □

Nothing can be said if $q = 1$.

**Remark 5.20** The previous theorem has another proof, which emphasizes the speed at which a sequence converges to 0 or $+\infty$. Take for example the case $q < 1$. The definition of limit tells that for all $r$ with $q < r < 1$, if one puts $\varepsilon = r - q$ there is a $n_\varepsilon \geq n_0$ such that

$$\frac{a_{n+1}}{a_n} < r \quad \text{that is,} \quad a_{n+1} < r a_n$$

for each $n > n_\varepsilon$. Repeating the argument leads to

$$a_{n+1} < r a_n < r^2 a_{n-1} < \ldots < r^{n-n_\varepsilon} a_{n_\varepsilon+1} \tag{5.16}$$

(a precise proof of which requires the Principle of Induction; see Appendix A.1, p. 430). The First comparison test and the limit behaviour of the geometric sequence (Example 5.18 i)) allow to conclude. Formula (5.16) shows that the smaller $q$ is, the faster the sequence $\{a_n\}$ goes to 0.
Similar considerations hold when $q > 1$. □

At last we consider a few significant sequences converging to $+\infty$. We compare their limit behaviour using Definition 5.10. To be precise we examine the sequences

$$\boxed{\log n\,,\; n^\alpha\,,\; q^n\,,\; n!\,,\; n^n \quad (\alpha > 0,\; q > 1)}$$

and show that each sequence is infinite of order bigger than the one preceding it. Comparing the first two is immediate, for the Substitution theorem and (5.6) c) yield $\log n = o(n^\alpha)$ for $n \to \infty$.
The remaining cases are tackled by applying the Ratio test 5.19 to the quotient of two nearby sequences. Precisely, let us set $a_n = \dfrac{n^\alpha}{q^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^\alpha}{q^{n+1}}\,\frac{q^n}{n^\alpha} = \left(\frac{n+1}{n}\right)^\alpha \frac{1}{q} \quad \to \quad \frac{1}{q} < 1\,, \qquad n \to \infty.$$

Thus $\lim_{n\to\infty} a_n = 0$, or $n^\alpha = o(q^n)$ for $n \to \infty$.