6

Differential calculus

The precise definition of the notion of derivative, studying a function's differentiability and computing its successive derivatives, the use of derivatives to analyse the local and global behaviours of functions are all constituents of Differential Calculus.

6.1 The derivative

We start by defining the derivative of a function.

Let $f : \text{dom } f \subseteq \mathbb{R} \to \mathbb{R}$ be a real function of one real variable, take $x_0 \in \text{dom } f$ and suppose f is defined in a neighbourhood $I_r(x_0)$ of x_0. With $x \in I_r(x_0)$, $x \neq x_0$ fixed, denote by

$$\Delta x = x - x_0$$

the (positive or negative) **increment of the independent variable between** x_0 **and** x, and by

$$\Delta f = f(x) - f(x_0)$$

the corresponding **increment of the dependent variable**. Note that $x = x_0 + \Delta x$, $f(x) = f(x_0) + \Delta f$.

The ratio

$$\boxed{\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}$$

is called **difference quotient of f between x_0 and x.**

In this manner Δf represents the *absolute increment* of the dependent variable f when passing from x_0 to $x_0 + \Delta x$, whereas the difference quotient detects the *rate of increment* (while $\Delta f / f$ is the *relative increment*). Multiplying the difference quotient by 100 we obtain the so-called *percentage increment*. Suppose a rise by $\Delta x = 0.2$ of the variable x prompts an increment $\Delta f = 0.06$ of f; the difference quotient $\frac{\Delta f}{\Delta x}$ equals $0.3 = \frac{30}{100}$, corresponding to a 30% increase.

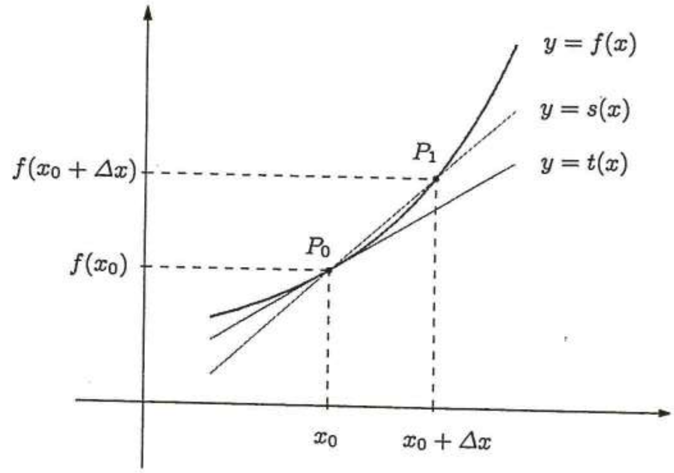

Figure 6.1. Secant and tangent lines to the graph of f at P_0

Graphically, the difference quotient between x_0 and a point x_1 around x_0 is the slope of the straight line s passing through $P_0 = (x_0, f(x_0))$ and $P_1 = (x_1, f(x_1))$, points that belong to the graph of the function; this line is called **secant** of the graph of f at P_0 and P_1 (Fig. 6.1). Putting $\Delta x = x_1 - x_0$ and $\Delta f = f(x_1) - f(x_0)$, the equation of the secant line reads

$$y = s(x) = f(x_0) + \frac{\Delta f}{\Delta x}(x - x_0), \qquad x \in \mathbb{R}. \tag{6.1}$$

A typical application of the difference quotient comes from physics. Let M be a point-particle moving along a straight line; call $s = s(t)$ the x-coordinate of the position of M at time t, with respect to a reference point O. Between the instants t_0 and $t_1 = t_0 + \Delta t$, the particle changes position by $\Delta s = s(t_1) - s(t_0)$. The difference quotient $\frac{\Delta s}{\Delta t}$ represents the *average velocity* of the particle in the given interval of time.

How does the difference quotient change, as Δx approaches 0? This is answered by the following notion.

Definition 6.1 *A map f defined on a neighbourhood of $x_0 \in \mathbb{R}$ is called* **differentiable at** x_0 *if the limit of the difference quotient* $\frac{\Delta f}{\Delta x}$ *between x_0 and x exists and is finite, as x approaches x_0. The real number*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called (**first**) **derivative of** f **at** x_0.

The derivative at x_0 is variously denoted, for instance also by

$$y'(x_0), \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(x_0), \qquad \mathrm{D}f(x_0).$$

The first symbol goes back to Newton, the second is associated to Leibniz.

From the geometric point of view $f'(x_0)$ is the slope of the **tangent line** at $P_0 = \big(x_0, f(x_0)\big)$ to the graph of f: such line t is obtained as the limiting position of the secant s at P_0 and $P = \big(x, f(x)\big)$, when P approaches P_0. From (6.1) and the previous definition we have

$$\boxed{y = t(x) = f(x_0) + f'(x_0)(x - x_0), \qquad x \in \mathbb{R}.}$$

In the physical example given above, the derivative $v(t_0) = s'(t_0) = \lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t}$ is the instantaneous *velocity* of the particle M at time t_0.

Let

$$\operatorname{dom} f' = \{x \in \operatorname{dom} f \ : \ f \text{ is differentiable at } x\}$$

and define the function $f' : \operatorname{dom} f' \subseteq \mathbb{R} \to \mathbb{R}$, $f' : x \mapsto f'(x)$ mapping $x \in \operatorname{dom} f'$ to the value of the derivative of f at x. This map is called **(first) derivative of** f.

Definition 6.2 *Let I be a subset of* $\operatorname{dom} f$. *We say that f is* **differentiable on** I *(or* **in** I*) if f is differentiable at each point of I.*

A first yet significant property of differentiable maps is the following.

Proposition 6.3 *If f is differentiable at x_0, it is also continuous at x_0.*

Proof. Continuity at x_0 prescribes

$$\lim_{x \to x_0} f(x) = f(x_0), \qquad \text{that is} \qquad \lim_{x \to x_0} \big(f(x) - f(x_0)\big) = 0.$$

If f is differentiable at x_0, then

$$\begin{aligned}
\lim_{x \to x_0} \big(f(x) - f(x_0)\big) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\,(x - x_0) \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot 0 = 0.
\end{aligned}$$

□

Not all continuous maps at a point are differentiable though. Consider the map $f(x) = |x|$: it is *continuous* at the origin, yet the difference quotient between the origin and a point $x \neq 0$ is

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases} \tag{6.2}$$

so the limit for $x \to 0$ does not exist. Otherwise said, f *is not differentiable* at the origin. This particular example shows that the implication of Proposition 6.3 can not be reversed: differentiability is thus a *stronger* property than continuity, an aspect to which Sect. 6.3 is entirely devoted.

6.2 Derivatives of the elementary functions. Rules of differentiation

We begin by tackling the issue of differentiability for elementary functions using Definition 6.1.

i) Consider the affine map $f(x) = ax + b$, and let $x_0 \in \mathbb{R}$ be arbitrary. Then

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\big(a(x_0 + \Delta x) + b\big) - (ax_0 + b)}{\Delta x} = \lim_{\Delta x \to 0} a = a,$$

in agreement with the fact that the graph of f is a straight line of slope a. The derivative of $f(x) = ax + b$ is then the constant map $f'(x) = a$.

In particular if f is constant ($a = 0$), its derivative is identically zero.

ii) Take $f(x) = x^2$ and $x_0 \in \mathbb{R}$. Since

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = \lim_{\Delta x \to 0} (2x_0 + \Delta x) = 2x_0,$$

the derivative of $f(x) = x^2$ is the function $f'(x) = 2x$.

iii) Now let $f(x) = x^n$ with $n \in \mathbb{N}$. The binomial formula (1.13) yields

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x_0^n + nx_0^{n-1}\Delta x + \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^k - x_0^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(nx_0^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^{k-1} \right) = nx_0^{n-1}. \end{aligned}$$

for all $x_0 \in \mathbb{R}$. Therefore, $f'(x) = nx^{n-1}$ is the derivative of $f(x) = x^n$.

iv) Even more generally, consider $f(x) = x^\alpha$ where $\alpha \in \mathbb{R}$, and let $x_0 \neq 0$ be a point of the domain. Then

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^\alpha - x_0^\alpha}{\Delta x} = \lim_{\Delta x \to 0} \frac{x_0^\alpha \left[\left(1 + \frac{\Delta x}{x_0}\right)^\alpha - 1\right]}{\Delta x}$$
$$= x_0^{\alpha-1} \lim_{\Delta x \to 0} \frac{\left(1 + \frac{\Delta x}{x_0}\right)^\alpha - 1}{\frac{\Delta x}{x_0}}.$$

Substituting $y = \frac{\Delta x}{x_0}$ brings the latter into the form of the fundamental limit (4.13), so

$$f'(x_0) = \alpha x_0^{\alpha-1}.$$

When $\alpha > 1$, f is differentiable at $x_0 = 0$ as well, and $f'(0) = 0$. The function $f(x) = x^\alpha$ is thus differentiable at all points where the expression $x^{\alpha-1}$ is well defined; its derivative is $f'(x) = \alpha x^{\alpha-1}$.

For example $f(x) = \sqrt{x} = x^{1/2}$, defined on $[0, +\infty)$, is differentiable on $(0, +\infty)$ with derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$. The function $f(x) = \sqrt[3]{x^5} = x^{5/3}$ is defined on $\mathbb{R}$, where it is also differentiable, and $f'(x) = \frac{5}{3}x^{2/3} = \frac{5}{3}\sqrt[3]{x^2}$.

v) Now consider the trigonometric functions. Take $f(x) = \sin x$ and $x_0 \in \mathbb{R}$. Formula (2.14) gives

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x} = \lim_{\Delta x \to 0} \frac{2 \sin \frac{\Delta x}{2} \cos(x_0 + \frac{\Delta x}{2})}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \lim_{\Delta x \to 0} \cos\left(x_0 + \frac{\Delta x}{2}\right).$$

The limit (4.5) and the cosine's continuity tell

$$f'(x_0) = \cos x_0.$$

Hence the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$.

Using in a similar way formula (2.15), we can see that the derivative of $f(x) = \cos x$ is the function $f'(x) = -\sin x$.

vi) Eventually, consider the exponential function $f(x) = a^x$. By (4.12) we have

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{a^{x_0 + \Delta x} - a^{x_0}}{\Delta x} = a^{x_0} \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = a^{x_0} \log a,$$

showing that the derivative of $f(x) = a^x$ is $f'(x) = (\log a)a^x$.

As $\log \mathrm{e} = 1$, the derivative of $f(x) = \mathrm{e}^x$ is $f'(x) = \mathrm{e}^x = f(x)$, whence the derivative f' coincides at each point with the function f itself. This is a crucial fact, and a reason for choosing e as privileged base for the exponential map.

We next discuss differentiability in terms of operations (algebraic operations, composition, inversion) on functions. We shall establish certain *differentiation rules* to compute derivatives of functions that are built from the elementary ones, without resorting to the definition each time. The proofs may be found in Appendix A.4.1, p. 449.

Theorem 6.4 (Algebraic operations) *Let $f(x), g(x)$ be differentiable maps at $x_0 \in \mathbb{R}$. Then the maps $f(x) \pm g(x)$, $f(x)g(x)$ and, if $g(x_0) \neq 0$, $\frac{f(x)}{g(x)}$ are differentiable at x_0. To be precise,*

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0), \tag{6.3}$$

$$(f\,g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0), \tag{6.4}$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}. \tag{6.5}$$

Corollary 6.5 (Linearity of the derivative) *If $f(x)$ and $g(x)$ are differentiable at $x_0 \in \mathbb{R}$, the map $\alpha f(x) + \beta g(x)$ is differentiable at x_0 for any $\alpha, \beta \in \mathbb{R}$ and*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0). \tag{6.6}$$

Proof. Consider (6.4) and recall that differentiating a constant gives zero; then $(\alpha f)'(x_0) = \alpha f'(x_0)$ and $(\beta g)'(x_0) = \beta g'(x_0)$ follow. The rest is a consequence of (6.3). □

Examples 6.6

i) To differentiate a polynomial, we use the fact that $D\,x^n = nx^{n-1}$ and apply the corollary repeatedly. So, $f(x) = 3x^5 - 2x^4 - x^3 + 3x^2 - 5x + 2$ differentiates to

$$f'(x) = 3 \cdot 5x^4 - 2 \cdot 4x^3 - 3x^2 + 3 \cdot 2x - 5 = 15x^4 - 8x^3 - 3x^2 + 6x - 5.$$

ii) For rational functions, we compute the numerator and denominator's derivatives and then employ rule (6.5), to the effect that

$$f(x) = \frac{x^2 - 3x + 1}{2x - 1}$$

has derivative

$$f'(x) = \frac{(2x-3)(2x-1) - (x^2 - 3x + 1)2}{(2x-1)^2} = \frac{2x^2 - 2x + 1}{4x^2 - 4x + 1}.$$

iii) Consider $f(x) = x^3 \sin x$. The product rule (6.4) together with $(\sin x)' = \cos x$ yield

$$f'(x) = 3x^2 \sin x + x^3 \cos x.$$

iv) The function

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

can be differentiated with (6.5)

$$f'(x) = \frac{\cos x\, \cos x - \sin x\,(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x.$$

Another possibility is to use $\cos^2 x + \sin^2 x = 1$ to obtain

$$f'(x) = \frac{1}{\cos^2 x}.$$

□

Theorem 6.7 (**"Chain rule"**) *Let $f(x)$ be differentiable at $x_0 \in \mathbb{R}$ and $g(y)$ a differentiable map at $y_0 = f(x_0)$. Then the composition $g \circ f(x) = g\big(f(x)\big)$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0) = g'\big(f(x_0)\big) f'(x_0). \tag{6.7}$$

Examples 6.8

i) The map $h(x) = \sqrt{1-x^2}$ is the composite of $f(x) = 1 - x^2$, whose derivative is $f'(x) = -2x$, and $g(y) = \sqrt{y}$, for which $g'(y) = \dfrac{1}{2\sqrt{y}}$. Then (6.7) directly gives

$$h'(x) = \frac{1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

ii) The function $h(x) = e^{\cos 3x}$ is composed by $f(x) = \cos 3x$, $g(y) = e^y$. But $f(x)$ is in turn the composite of $\varphi(x) = 3x$ and $\psi(y) = \cos y$; thus (6.7) tells $f'(x) = -3 \sin 3x$. On the other hand $g'(y) = e^y$. Using (6.7) once again we conclude

$$h'(x) = -3e^{\cos 3x} \sin 3x.$$

□

Theorem 6.9 (**Derivative of the inverse function**) *Suppose $f(x)$ is a continuous, invertible map on a neighbourhood of $x_0 \in \mathbb{R}$, and differentiable at x_0, with $f'(x_0) \neq 0$. Then the inverse map $f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}. \tag{6.8}$$

Examples 6.10

i) The function $y = f(x) = \tan x$ has derivative $f'(x) = 1 + \tan^2 x$ and inverse $x = f^{-1}(y) = \arctan y$. By (6.8)

$$(f^{-1})'(y) = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}.$$

Setting for simplicity $f^{-1} = g$ and denoting the independent variable with x, the derivative of $g(x) = \arctan x$ is the function $g'(x) = \dfrac{1}{1+x^2}$.

ii) We are by now acquainted with the function $y = f(x) = \sin x$: it is invertible on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, namely $x = f^{-1}(y) = \arcsin y$. Moreover, f differentiates to $f'(x) = \cos x$. Using $\cos^2 x + \sin^2 x = 1$, and taking into account that on that interval $\cos x \geq 0$, one can write the derivative of f in the equivalent form $f'(x) = \sqrt{1-\sin^2 x}$. Now (6.8) yields

$$(f^{-1})'(y) = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.$$

Put once again $f^{-1} = g$ and change names to the variables: the derivative of $g(x) = \arcsin x$ is $g'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

In similar fashion $g(x) = \arccos x$ differentiates to $g'(x) = -\dfrac{1}{\sqrt{1-x^2}}$.

iii) Consider $y = f(x) = a^x$. It has derivative $f'(x) = (\log a)a^x$ and inverse $x = f^{-1}(y) = \log_a y$. The usual (6.8) gives

$$(f^{-1})'(y) = \frac{1}{(\log a)a^x} = \frac{1}{(\log a)y}.$$

Defining $f^{-1} = g$ and renaming x the independent variable gives $g'(x) = \dfrac{1}{(\log a)x}$ as derivative of $g(x) = \log_a x$ $(x > 0)$.

Take now $h(x) = \log_a(-x)$ (with $x < 0$), composition of $x \mapsto -x$ and $g(y)$: then $h'(x) = \dfrac{1}{(\log a)(-x)}(-1) = \dfrac{1}{(\log a)x}$. Putting all together shows that $g(x) = \log_a |x|$ $(x \neq 0)$ has derivative $g'(x) = \dfrac{1}{(\log a)x}$.

With the choice of base $a = \mathrm{e}$ the derivative of $g(x) = \log |x|$ is $g'(x) = \dfrac{1}{x}$. □

Remark 6.11 Let $f(x)$ be differentiable and strictly positive on an interval I. Due to the previous result and the Chain rule, the derivative of the composite map $g(x) = \log f(x)$ is

$$g'(x) = \frac{f'(x)}{f(x)}.$$

The expression $\dfrac{f'}{f}$ is said **logarithmic derivative** of the map f. □

The section ends with a useful corollary to the Chain rule 6.7.

Property 6.12 *If f is an even (or odd) differentiable function on all its domain, the derivative f' is odd (resp. even).*

Proof. Since f is even, $f(-x) = f(x)$ for any $x \in \operatorname{dom} f$. Let us differentiate both sides. As $f(-x)$ is the composition of $x \mapsto -x$ and $y \mapsto f(y)$, its derivative reads $-f'(-x)$. Then $f'(-x) = -f'(x)$ for all $x \in \operatorname{dom} f$, so f' is odd. Similarly if f is odd. □

We reckon it could be useful to collect the derivatives of the main elementary functions in one table, for reference.

$$\mathrm{D}\, x^\alpha = \alpha x^{\alpha-1} \qquad (\forall \alpha \in \mathbb{R})$$

$$\mathrm{D}\, \sin x = \cos x$$

$$\mathrm{D}\, \cos x = -\sin x$$

$$\mathrm{D}\, \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\mathrm{D}\, \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arctan x = \frac{1}{1+x^2}$$

$$\mathrm{D}\, a^x = (\log a)\, a^x \qquad \text{in particular,} \quad \mathrm{D}\, \mathrm{e}^x = \mathrm{e}^x$$

$$\mathrm{D}\, \log_a |x| = \frac{1}{(\log a)\, x} \qquad \text{in particular,} \quad \mathrm{D}\, \log |x| = \frac{1}{x}$$

6.3 Where differentiability fails

It was noted earlier that the function $f(x) = |x|$ is continuous but not differentiable at the origin. At each other point of the real line f is differentiable, for it coincides with the line $y = x$ when $x > 0$, and with $y = -x$ for $x < 0$. Therefore $f'(x) = +1$

for $x > 0$ and $f'(x) = -1$ on $x < 0$. The reader will recall the sign function (Example 2.1 iv)), for which

$$\mathrm{D}\,|x| = \mathrm{sign}(x), \qquad \text{for all } x \neq 0.$$

The origin is an *isolated point of non-differentiability* for $y = |x|$.

Returning to the expression (6.2) for the difference quotient at the origin, we observe that the one-sided limits exist and are finite:

$$\lim_{x \to 0+} \frac{\Delta f}{\Delta x} = 1, \qquad \lim_{x \to 0-} \frac{\Delta f}{\Delta x} = -1.$$

This fact suggests us to introduce the following notion.

Definition 6.13 *Suppose f is defined on a right neighbourhood of $x_0 \in \mathbb{R}$. It is called* **differentiable on the right at** x_0 *if the right limit of the difference quotient $\frac{\Delta f}{\Delta x}$ between x_0 and x exists finite, for x approaching x_0. The real number*

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is the **right** (*or* **backward**) **derivative of** f **at** x_0. *Similarly it goes for the* **left** (*or* **forward**) **derivative** $f'_-(x_0)$.

If f is defined only on a right (resp. left) neighbourhood of x_0 and is differentiable on the right (resp. the left) at x_0, we shall simply say that f is differentiable at x_0, and write $f'(x_0) = f'_+(x_0)$ (resp. $f'(x_0) = f'_-(x_0)$).

From Proposition 3.24 the following criterion is immediate.

Property 6.14 *A map f defined around a point $x_0 \in \mathbb{R}$ is differentiable at x_0 if and only if it is differentiable on both sides at x_0 and the left and right derivatives coincide, in which case*

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

Instead, if f is differentiable at x_0 on the left and on the right, but the two derivatives are different (as for $f(x) = |x|$ at the origin), x_0 is called **corner (point)** for f (Fig. 6.2). The term originates in the geometric observation that the right derivative of f at x_0 represents the slope of the *right tangent* to the graph of f at $P_0 = (x_0, f(x_0))$, i.e., the limiting position of the secant through P_0 and $P = (x, f(x))$ as $x > x_0$ approaches x_0. In case the right and left tangent (similarly defined) do not coincide, they form an *angle* at P_0.

for $x > 0$ and $f'(x) = -1$ on $x < 0$. The reader will recall the sign function (Example 2.1 iv)), for which

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$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is the **right** (*or* **backward**) **derivative of** f **at** x_0. *Similarly it goes for the* **left** (*or* **forward**) **derivative** $f'_-(x_0)$.

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Instead, if f is differentiable at x_0 on the left and on the right, but the two derivatives are different (as for $f(x) = |x|$ at the origin), x_0 is called **corner (point)** for f (Fig. 6.2). The term originates in the geometric observation that the right derivative of f at x_0 represents the slope of the *right tangent* to the graph of f at $P_0 = (x_0, f(x_0))$, i.e., the limiting position of the secant through P_0 and $P = (x, f(x))$ as $x > x_0$ approaches x_0. In case the right and left tangent (similarly defined) do not coincide, they form an *angle* at P_0.

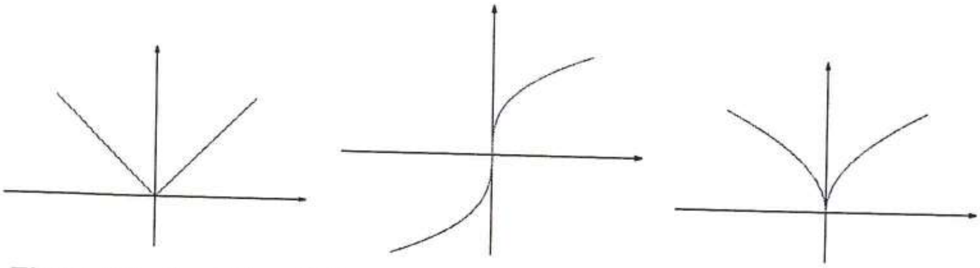


Figure 6.2. Non-differentiable maps: the origin is a corner point (left), a point with vertical tangent (middle), a cusp (right)

Other interesting cases occur when the right and left limits of the difference quotient of f at x_0 exist, but one at least is not finite. These will be still denoted by $f'_+(x_0)$ and $f'_-(x_0)$.

Precisely, if just one of $f'_+(x_0)$, $f'_-(x_0)$ is infinite, we still say that x_0 is a **corner point** for f.

If both $f'_+(x_0)$ and $f'_-(x_0)$ are infinite and with same sign (hence the limit of the difference quotient is $+\infty$ or $-\infty$), x_0 is a **point with vertical tangent** for f. This is the case for $f(x) = \sqrt[3]{x}$:

$$f'_\pm(0) = \lim_{x\to 0^\pm} \frac{\sqrt[3]{x}}{x} = \lim_{x\to 0^\pm} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

When $f'_+(x_0)$, $f'_-(x_0)$ are finite and have different signs, x_0 is called a **cusp (point)** of f. For instance the map $f(x) = \sqrt{|x|}$ has a cusp at the origin, for

$$f'_\pm(0) = \lim_{x\to 0^\pm} \frac{\sqrt{|x|}}{x} = \lim_{x\to 0^\pm} \frac{\sqrt{|x|}}{\operatorname{sign}(x)\,|x|} = \lim_{x\to 0^\pm} \frac{1}{\operatorname{sign}(x)\,\sqrt{|x|}} = \pm\infty.$$

Another criterion for differentiability at a point x_0 is up next. The proof is deferred to Sect. 6.11, for it relies on de l'Hôpital's Theorem.

Theorem 6.15 *Let f be continuous at x_0 and differentiable at all points $x \neq x_0$ in a neighbourhood of x_0. Then f is differentiable at x_0 provided that the limit of $f'(x)$ for $x \to x_0$ exists finite. If so,*

$$f'(x_0) = \lim_{x\to x_0} f'(x).$$

Example 6.16

We take the function

$$f(x) = \begin{cases} a\sin 2x - 4 & \text{if } x < 0, \\ b(x-1) + e^x & \text{if } x \geq 0, \end{cases}$$

and ask ourselves whether there are real numbers a and b rendering f differentiable at the origin. The continuity at the origin (recall: differentiable implies continuous) forces the two values

$$\lim_{x\to 0^-} f(x) = -4, \qquad \lim_{x\to 0^+} f(x) = f(0) = -b+1$$

to agree, hence $b = 5$. With b fixed, we may impose the equality of the right and left limits of $f'(x)$ for $x \to 0$, to the effect that $f'(x)$ admits finite limit for $x \to 0$. Then we use Theorem 6.15, which prescribes that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2a\cos 2x = 2a, \qquad \text{and} \qquad \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (5+e^x) = 6$$

are the same, so $a = 3$. □

Remark 6.17 In using Theorem 6.15 one should not forget to impose continuity at the point x_0. The mere existence of the limit for f' is not enough to guarantee f will be differentiable at x_0. For example, $f(x) = x + \operatorname{sign} x$ is differentiable at every $x \neq 0$: since $f'(x) = 1$, it necessarily follows $\lim_{x\to 0} f'(x) = 1$. The function is nonetheless not differentiable, because not continuous, at $x = 0$. □

6.4 Extrema and critical points

Definition 6.18 *One calls* $x_0 \in \operatorname{dom} f$ *a* **relative** (*or* **local**) **maximum point for** f *if there is a neighbourhood* $I_r(x_0)$ *of* x_0 *such that*

$$\forall x \in I_r(x_0) \cap \operatorname{dom} f, \qquad f(x) \le f(x_0).$$

Then $f(x_0)$ *is a* **relative** (*or* **local**) **maximum of** f.
One calls x_0 *an* **absolute maximum point** (*or* **global maximum point**) **for** f *if*

$$\forall x \in \operatorname{dom} f, \qquad f(x) \le f(x_0),$$

and $f(x_0)$ *becomes the* (**absolute**) **maximum of** f. *In either case, the maximum is said* **strict** *if* $f(x) < f(x_0)$ *when* $x \neq x_0$.

Exchanging the symbols $\le$ with $\ge$ one obtains the definitions of **relative** and **absolute minimum point**. A minimum or maximum point shall be referred to generically as an **extremum** (**point**) of f.

Examples 6.19

i) The parabola $f(x) = 1+2x-x^2 = 2-(x-1)^2$ has a strict absolute maximum point at $x_0 = 1$, and 2 is the function's absolute maximum. Notice the derivative $f'(x) = 2(1-x)$ is zero at that point. There are no minimum points (relative or absolute).

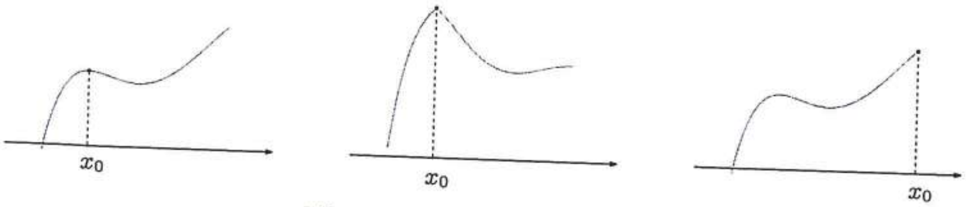


Figure 6.3. Types of maxima

ii) For $g(x) = \arcsin x$ (see Fig. 2.24), $x_0 = 1$ is a strict absolute maximum point, with maximum value $\frac{\pi}{2}$. The point $x_1 = -1$ is a strict absolute minimum, with value $-\frac{\pi}{2}$. At these extrema g is not differentiable. □

We are interested in finding the extremum points of a given function. Provided the latter is differentiable, it might be useful to look for the points where the first derivative vanishes.

Definition 6.20 *A* **critical point** (*or* **stationary point**) *of f is a point x_0 at which f is differentiable with derivative $f'(x_0) = 0$.*

The tangent at a critical point is horizontal.

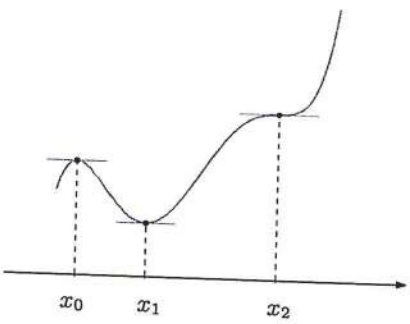


Figure 6.4. Types of critical points

Theorem 6.21 (Fermat) *Suppose f is defined in a full neighbourhood of a point x_0 and differentiable at x_0. If x_0 is an extremum point, then it is critical for f, i.e.,*

$$f'(x_0) = 0.$$

6.5 Theorems of Rolle, Lagrange, and Cauchy

We begin this section by presenting two theorems, Rolle's Theorem and Lagrange's or Mean Value Theorem, that are fundamental for the study of differentiable maps on an interval.

Theorem 6.22 (Rolle) *Let f be a function defined on a closed bounded interval $[a,b]$, continuous on $[a,b]$ and differentiable on (a,b) (at least). If $f(a) = f(b)$, there exists an $x_0 \in (a,b)$ such that*

$$f'(x_0) = 0.$$

In other words, f admits at least one critical point in (a,b).

Proof. By the Theorem of Weierstrass the range $f([a,b])$ is the closed interval $[m, M]$ bounded by the minimum and maximum values m, M of the map:

$$m = \min_{x\in[a,b]} f(x) = f(x_m), \qquad M = \max_{x\in[a,b]} f(x) = f(x_M),$$

for suitable $x_m, x_M \in [a,b]$.
In case $m = M$, f is constant on $[a,b]$, so in particular $f'(x) = 0$ for any $x \in (a,b)$ and the theorem follows.
Suppose then $m < M$. Since $m \le f(a) = f(b) \le M$, one of the strict inequalities $f(a) = f(b) < M$, $m < f(a) = f(b)$ will hold.
If $f(a) = f(b) < M$, the absolute maximum point x_M cannot be a nor b; thus, $x_M \in (a,b)$ is an interior extremum point at which f is differentiable. By Fermat's Theorem 6.21 we have that $x_M = x_0$ is a critical point.
If $m < f(a) = f(b)$, one proves analogously that x_m is the critical point x_0 of the claim. □

The theorem proves the existence of one critical point in (a,b); Fig. 6.5 shows that there could actually be more.

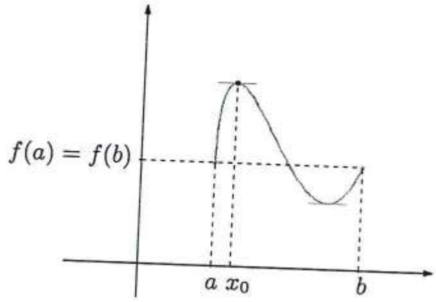


Figure 6.5. Rolle's Theorem

Proof. To fix ideas, assume x_0 is a relative maximum point and that $I_r(x_0)$ is a neighbourhood where $f(x) \le f(x_0)$ for all $x \in I_r(x_0)$. On such neighbourhood then $\Delta f = f(x) - f(x_0) \le 0$.

If $x > x_0$, hence $\Delta x = x - x_0 > 0$, the difference quotient $\frac{\Delta f}{\Delta x}$ is non-positive. Corollary 4.3 implies

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \le 0.$$

Vice versa, if $x < x_0$, i.e., $\Delta x < 0$, then $\frac{\Delta f}{\Delta x}$ is non-negative, so

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \ge 0.$$

By Property 6.14,

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0},$$

so $f'(x_0)$ is simultaneously ≤ 0 and ≥ 0, hence zero.
A similar argument holds for relative minima. □

Fermat's Theorem 6.21 ensures that the extremum points of a differentiable map which belong to the *interior* of the domain should be searched for among critical points.

A function can nevertheless have critical points that are not extrema, as in Fig. 6.4. The map $f(x) = x^3$ has the origin as a critical point ($f'(x) = 3x^2 = 0$ if and only if $x = 0$), but admits no extremum since it is strictly increasing on the whole $\mathbb{R}$.

At the same time though, a function may have non-critical extremum point (Fig. 6.3); this happens when a function is not differentiable at an extremum that lies inside the domain (e.g. $f(x) = |x|$, whose absolute minimum is attained at the origin), or when the extremum point is on the boundary (as in Example 6.19 ii)). The upshot is that in order to find *all* extrema of a function, browsing through the critical points might not be sufficient.

To summarise, extremum points are contained among the points of the domain at which either

i) the first derivative vanishes,
ii) or the function is not differentiable,
iii) or among the domain's boundary points (inside $\mathbb{R}$).

Theorem 6.23 (**Mean Value Theorem** or **Lagrange Theorem**) *Let f be defined on the closed and bounded interval $[a,b]$, continuous on $[a,b]$ and differentiable (at least) on (a,b). Then there is a point $x_0 \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(x_0). \tag{6.9}$$

Every such point x_0 we shall call **Lagrange point for f in (a,b)**.

Proof. Introduce an auxiliary map

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$$

defined on $[a,b]$. It is continuous on $[a,b]$ and differentiable on (a,b), as difference of f and an affine map, which is differentiable on all of $\mathbb{R}$. Note

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

It is easily seen that

$$g(a) = f(a), \qquad g(b) = f(a),$$

so Rolle's Theorem applies to g, with the consequence that there is a point $x_0 \in (a,b)$ satisfying

$$g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0.$$

But this is exactly (6.9). □

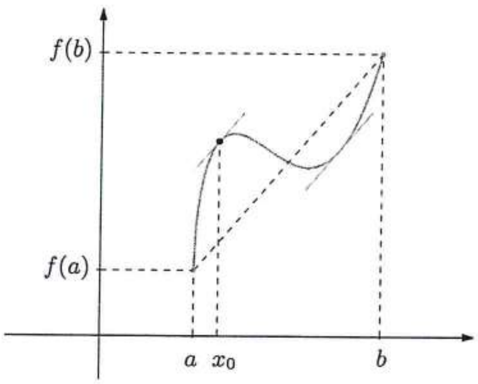


Figure 6.6. Lagrange point for f in (a,b)

The meaning of the Mean Value Theorem is clarified in Fig. 6.6. At each Lagrange point, the tangent to the graph of f is *parallel* to the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.

Example 6.24

Consider $f(x) = 1 + x + \sqrt{1-x^2}$, a continuous map on its domain $[-1, 1]$ as composite of elementary continuous functions. It is also differentiable on the open interval $(-1, 1)$ (not at the end-points), in fact

$$f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}.$$

Thus f fulfills the Mean Value Theorem's hypotheses, and must admit a Lagrange point in $(-1, 1)$. Now (6.9) becomes

$$1 = \frac{f(1) - f(-1)}{1 - (-1)} = f'(x_0) = 1 - \frac{x_0}{\sqrt{1-x_0^2}},$$

satisfied by $x_0 = 0$. □

The following result is a generalisation of the Mean Value Theorem 6.23 (which is recovered by $g(x) = x$ in its statement). It will be useful during the proofs of de l'Hôpital's Theorem 6.41 and Taylor's formula with Lagrange's remainder (Theorem 7.2).

Theorem 6.25 (Cauchy) *Let f and g be maps defined on the closed, bounded interval $[a, b]$, continuous on $[a, b]$ and differentiable (at least) on (a, b). Suppose $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $x_0 \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}. \tag{6.10}$$

Proof. Note first that $g(a) \neq g(b)$, otherwise Rolle's Theorem would have $g'(x)$ vanish somewhere in (a, b), against the assumption.

Take the function

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$$

defined on $[a, b]$. It is continuous on $[a, b]$ and differentiable on the open interval (a, b), because difference of maps with those properties. Moreover

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)}\, g'(x).$$

As

$$h(a) = f(a), \qquad h(b) = f(a),$$

the map h satisfies Rolle's Theorem, so there must be a point $x_0 \in (a,b)$ with

$$h'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(x_0) = 0,$$

which is exactly (6.10). □

6.6 First and second finite increment formulas

We shall discuss a couple of useful relations to represent how a function varies when passing from one point to another of its domain.

Let us begin by assuming f is differentiable at x_0. By definition

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0),$$

that is to say

$$\lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \right) = \lim_{x\to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{x-x_0} = 0.$$

Using the Landau symbols of Sect. 5.1, this becomes

$$f(x)-f(x_0)-f'(x_0)(x-x_0) = o(x-x_0), \quad x \to x_0.$$

An equivalent formulation is

$$\boxed{f(x)-f(x_0) = f'(x_0)(x-x_0) + o(x-x_0), \quad x \to x_0,} \tag{6.11}$$

or

$$\boxed{\Delta f = f'(x_0)\Delta x + o(\Delta x), \quad \Delta x \to 0,} \tag{6.12}$$

by putting $\Delta x = x - x_0$ and $\Delta f = f(x) - f(x_0)$.
Equations (6.11)-(6.12) are equivalent writings of what we call the **first formula of the finite increment**, the geometric interpretation of which can be found in Fig. 6.7. It tells that if $f'(x_0) \neq 0$, the increment Δf, corresponding to a change Δx, is proportional to Δx itself, if one disregards an infinitesimal which is negligible with respect to Δx. For Δx small enough, in practice, Δf can be treated as $f'(x_0)\Delta x$.

Now take f continuous on an interval I of $\mathbb{R}$ and differentiable on the interior points. Fix $x_1 < x_2$ in I and note that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore f, restricted to $[x_1, x_2]$, satisfies the Mean Value Theorem, so there is $\bar{x} \in (x_1, x_2)$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\bar{x}),$$

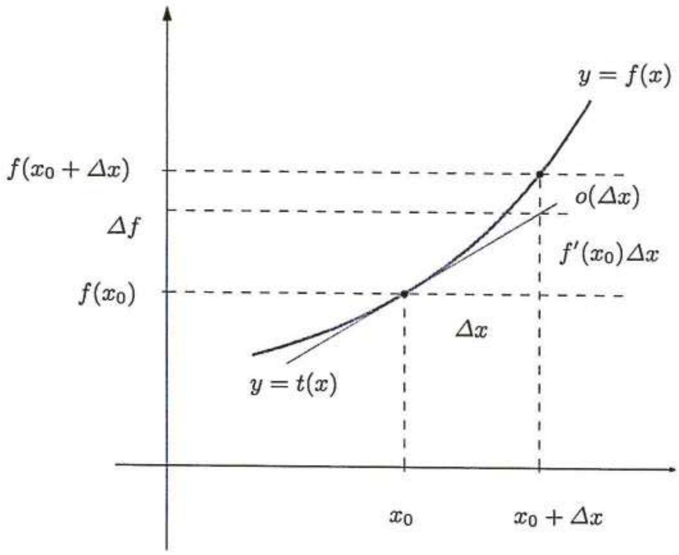


Figure 6.7. First formula of the finite increment

that is, a point $\bar{x} \in (x_1, x_2)$ with

$$\boxed{f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1).} \tag{6.13}$$

We shall refer to this relation as the **second formula of the finite increment.** It has to be noted that the point $\bar{x}$ depends upon the choice of x_1 and x_2, albeit this dependency is in general not explicit. The formula's relevance derives from the possibility of gaining information about the increment $f(x_2) - f(x_1)$ from the behaviour of f' on the interval $[x_1, x_2]$.

The second formula of the finite increment may be used to describe the local behaviour of a map in the neighbourhood of a certain x_0 with more precision than that permitted by the first formula. Suppose f is continuous at x_0 and differentiable around x_0 except possibly at the point itself. If x is a point in the neighbourhood of x_0, (6.13) can be applied to the interval bounded by x_0 and x, to the effect that

$$\boxed{\Delta f = f'(\bar{x})\Delta x,} \tag{6.14}$$

where $\bar{x}$ lies between x_0 and x. This alternative formulation of (6.13) expresses the increment of the dependent variable Δf as if it were a multiple of Δx; at closer look though, one realises that the proportionality coefficient, i.e., the derivative evaluated at a point near x_0, depends upon Δx (and on x_0), besides being usually not known.

A further application of (6.13) is described in the next result. This will be useful later.

Property 6.26 *A function defined on a real interval I and everywhere differentiable is constant on I if and only if its first derivative vanishes identically.*

Proof. Let f be the map. Suppose first f is constant, therefore for every $x_0 \in I$, the difference quotient $\dfrac{f(x)-f(x_0)}{x-x_0}$, with $x \in I$, $x \neq x_0$, is zero. Then $f'(x_0) = 0$ by definition of derivative.
Vice versa, suppose f has zero derivative on I and let us prove that f is constant on I. This would be equivalent to demanding

$$f(x_1) = f(x_2), \qquad \forall x_1, x_2 \in I.$$

Take $x_1, x_2 \in I$ and use formula (6.13) on f. For a suitable $\bar{x}$ between x_1, x_2, we have

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1) = 0,$$

thus $f(x_1) = f(x_2)$. □

6.7 Monotone maps

In the light of the results on differentiability, we tackle the issue of monotonicity.

Theorem 6.27 *Let I be an interval upon which the map f is differentiable. Then:*

a) If f is increasing on I, then $f'(x) \geq 0$ for all $x \in I$.

b1) If $f'(x) \geq 0$ for any $x \in I$, then f is increasing on I;

b2) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.

Proof. Let us prove claim *a*). Suppose f increasing on I and consider an interior point x_0 of I. For all $x \in I$ such that $x < x_0$, we have

$$f(x) - f(x_0) \leq 0 \qquad \text{and} \qquad x - x_0 < 0.$$

Thus, the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x is non-negative. On the other hand, for any $x \in I$ with $x > x_0$,

$$f(x) - f(x_0) \geq 0 \qquad \text{and} \qquad x - x_0 > 0.$$

Here too the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x is positive or zero. Altogether,

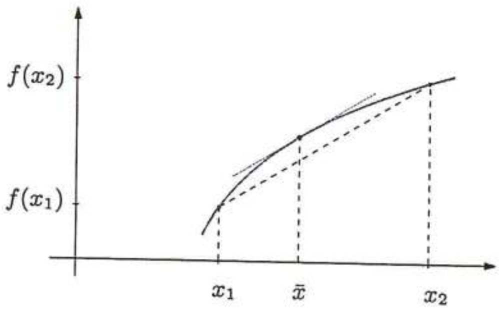


Figure 6.8. Proof of Theorem 6.27, *b*)

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} \geq 0, \qquad \forall x \neq x_0;$$

Corollary 4.3 on

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

yields $f'(x_0) \geq 0$. As for the possible extremum points in I, we arrive at the same conclusion by considering one-sided limits of the difference quotient, which is always ≥ 0.

Now to the implications in parts *b*). Take f with $f'(x) \geq 0$ for all $x \in I$. The idea is to fix points $x_1 < x_2$ in I and prove that $f(x_1) \leq f(x_2)$. For that we use (6.13) and note that $f'(\bar{x}) \geq 0$ by assumption. But since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1) \geq 0,$$

proving *b*1). Considering f such that $f'(x) > 0$ for all $x \in I$ instead, (6.13) implies $f(x_2) - f(x_1) > 0$, hence also *b*2) holds. □

The theorem asserts that if f is differentiable on I, the following logic equivalence holds:

$$f'(x) \geq 0, \quad \forall x \in I \quad \Longleftrightarrow \quad f \text{ is increasing on } I.$$

Furthermore,

$$f'(x) > 0, \quad \forall x \in I \quad \Longrightarrow \quad f \text{ is strictly increasing on } I.$$

The latter implication is not reversible: f *strictly* increasing on I does not imply $f'(x) > 0$ for all $x \in I$. We have elsewhere observed that $f(x) = x^3$ is everywhere strictly increasing, despite having vanishing derivative at the origin.

A similar statement to the above holds if we change the word 'increasing' with 'decreasing' and the symbols $\geq$, $>$ with $\leq$, $<$.

Corollary 6.28 *Let f be differentiable on I and x_0 an interior critical point. If $f'(x) \geq 0$ at the left of x_0 and $f'(x) \leq 0$ at its right, then x_0 is a maximum point for f. Similarly, $f'(x) \leq 0$ at the left, and ≥ 0 at the right of x_0 implies x_0 is a minimum point.*

Theorem 6.27 and Corollary 6.28 justify the search for extrema among the zeroes of f', and explain why the derivative's sign affects monotonicity intervals.

Example 6.29

The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = xe^{2x}$ differentiates to $f'(x) = (2x+1)e^{2x}$, whence $x_0 = -\frac{1}{2}$ is the sole critical point. As $f'(x) > 0$ if and only if $x > -\frac{1}{2}$, $f(x_0)$ is an absolute minimum. The function is strictly decreasing on $(-\infty, -\frac{1}{2}]$ and strictly increasing on $[-\frac{1}{2}, +\infty)$. □

6.8 Higher-order derivatives

Let f be differentiable around x_0 and let its first derivative f' be also defined around x_0.

Definition 6.30 *If f' is a differentiable function at x_0, one says f* **is twice differentiable at** x_0. *The expression*

$$f''(x_0) = (f')'(x_0)$$

is called **second derivative of** f **at** x_0. *The* **second derivative of** f, *denoted f'', is the map associating to x the number $f''(x)$, provided the latter is defined.*

Other notations commonly used for the second derivative include

$$y''(x_0), \qquad \frac{d^2 f}{dx^2}(x_0), \qquad D^2 f(x_0).$$

The third derivative, where defined, is the derivative of the second derivative:

$$f'''(x_0) = (f'')'(x_0).$$

In general, for any $k \geq 1$, the **derivative of order** k **(kth derivative) of** f **at** x_0 is the first derivative, where defined, of the derivative of order $(k-1)$ of f at x_0:

$$\boxed{f^{(k)}(x_0) = (f^{(k-1)})'(x_0).}$$

Alternative symbols are:

$$y^{(k)}(x_0), \qquad \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x_0), \qquad \mathrm{D}^k f(x_0).$$

For conveniency one defines $f^{(0)}(x_0) = f(x_0)$ as well.

Examples 6.31

We compute the derivatives of all orders for three elementary functions.

i) Choose $n \in \mathbb{N}$ and consider $f(x) = x^n$. Then

$$\begin{aligned} f'(x) &= nx^{n-1} = \frac{n!}{(n-1)!}x^{n-1} \\ f''(x) &= n(n-1)x^{n-2} = \frac{n!}{(n-2)!}x^{n-2} \\ &\vdots \quad \vdots \\ f^{(n)}(x) &= n(n-1)\cdots 2\cdot 1\, x^{n-n} = n!\,. \end{aligned}$$

More concisely,

$$f^{(k)}(x) = \frac{n!}{(n-k)!}x^{n-k}$$

with $0 \le k \le n$. Furthermore, $f^{(n+1)}(x) = 0$ for any $x \in \mathbb{R}$ (the derivative of the constant function $f^{(n)}(x)$ is 0), and consequently all derivatives $f^{(k)}$ of order $k > n$ exist and vanish identically.

ii) The sine function $f(x) = \sin x$ satisfies $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f^{(4)}(x) = \sin x$. Successive derivatives of f clearly reproduce this cyclical pattern. The same phenomenon occurs for $y = \cos x$.

iii) Because $f(x) = \mathrm{e}^x$ differentiates to $f'(x) = \mathrm{e}^x$, it follows that $f^{(k)}(x) = \mathrm{e}^x$ for every $k \ge 0$, proving the remarkable fact that all higher-order derivatives of the exponential function are equal to e^x. □

A couple of definitions wrap up the section.

Definition 6.32 *A map f is* **of class** C^k *($k \ge 0$) on an interval I if f is differentiable k times everywhere on I and its kth derivative $f^{(k)}$ is continuous on I. The collection of all C^k maps on I is denoted by $C^k(I)$.*
A map f is **of class** C^∞ *on I if it is arbitrarily differentiable everywhere on I. One indicates by $C^\infty(I)$ the collection of such maps.*

In virtue of Proposition 6.3, if $f \in C^k(I)$ all derivatives of order smaller or equal than k are continuous on I. Similarly, if $f \in C^\infty(I)$, all its derivatives are continuous on I.

Moreover, the elementary functions are differentiable any number of times (so they are of class C^∞) at every interior point of their domains.

6.9 Convexity and inflection points

Let f be differentiable at the point x_0 of the domain. As customary, we indicate by $y = t(x) = f(x_0) + f'(x_0)(x - x_0)$ the equation of the tangent to the graph of f at x_0.

Definition 6.33 *The map f is* **convex at** x_0 *if there is a neighbourhood $I_r(x_0) \subseteq \operatorname{dom} f$ such that*

$$\forall x \in I_r(x_0), \qquad f(x) \geq t(x);$$

f is **strictly convex** *if $f(x) > t(x)$, $\forall x \neq x_0$.*

The definitions for **concave** and **strictly concave** functions are alike (just change $\geq, >$ to $\leq, <$).

What does this say geometrically? A map is convex at a point if around that point the graph lies 'above' the tangent line, concave if its graph is 'below' the tangent (Fig. 6.9).

Example 6.34

We claim that $f(x) = x^2$ is strictly convex at $x_0 = 1$. The tangent at the given point has equation

$$t(x) = 1 + 2(x - 1) = 2x - 1\,.$$

Since $f(x) > t(x)$ means $x^2 > 2x - 1$, hence $x^2 - 2x + 1 = (x - 1)^2 > 0$, t lies below the graph except at the touching point $x = 1$. □

Definition 6.35 *A differentiable map f on an interval I is* **convex on** I *if it is convex at each point of I.*

For understanding convexity, inflection points play a role reminiscent of extremum points for the study of monotone functions.

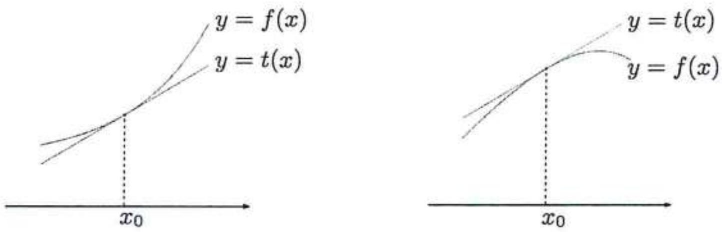


Figure 6.9. Strictly convex (left) and strictly concave (right) maps at x_0

Definition 6.36 *The point* x_0 *is an* **inflection point for** f *if there is a neighbourhood* $I_r(x_0) \subseteq \operatorname{dom} f$ *where one of the following conditions holds: either*

$$\forall x \in I_r(x_0), \qquad \begin{cases} \text{if } x < x_0, & f(x) \le t(x), \\ \text{if } x > x_0, & f(x) \ge t(x), \end{cases}$$

or

$$\forall x \in I_r(x_0), \qquad \begin{cases} \text{if } x < x_0, & f(x) \ge t(x), \\ \text{if } x > x_0, & f(x) \le t(x) \end{cases}$$

In the former case we speak of an **ascending** *inflection, in the latter the inflection is* **descending**.

In the plane, the graph of f 'cuts through' the inflectional tangent at an inflection point (Fig. 6.10).

The analysis of convexity and inflections of a function is helped a great deal by the next results.

Theorem 6.37 *Given a differentiable map* f *on the interval* I,

a) if f *is convex on* I*, then* f' *is increasing on* I.

b1) If f' *is increasing on* I*, then* f *is convex on* I;

b2) if f' *is strictly increasing on* I*, then* f *is strictly convex on* I.

Proof. See Appendix A.4.3, p. 455. □

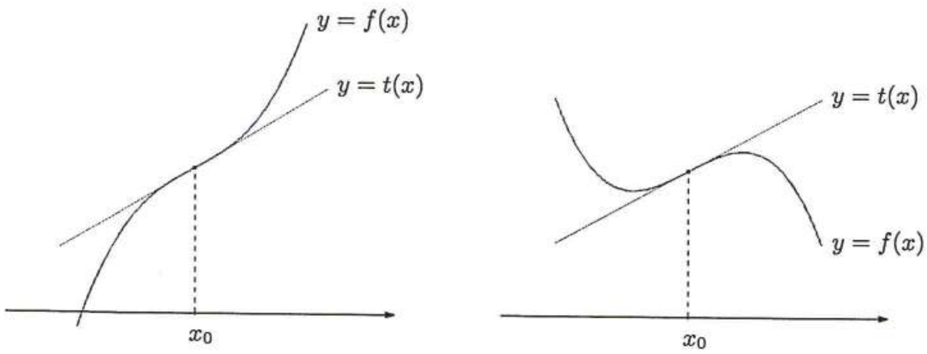


Figure 6.10. Ascending (left) and descending (right) inflections at x_0

Corollary 6.38 *If f is differentiable twice on I, then*

a) f convex on I implies $f''(x) \geq 0$ for all $x \in I$.

b1) $f''(x) \geq 0$ for all $x \in I$ implies f convex on I;

b2) $f''(x) > 0$ for all $x \in I$ implies f strictly convex on I.

Proof. This follows directly from Theorem 6.37 by applying Theorem 6.27 to the function f'. □

There is a second formulation for this, namely: under the same hypothesis, the following formulas are true:

$$\boxed{f''(x) \geq 0, \quad \forall x \in I \quad \Longleftrightarrow \quad f \text{ is convex on } I}$$

and

$$\boxed{f''(x) > 0, \quad \forall x \in I \quad \Longrightarrow \quad f \text{ is strictly convex on } I.}$$

Here, as in the characterisation of monotone functions, the last implication has no reverse. For instance, $f(x) = x^4$ is strictly convex on $\mathbb{R}$, but has vanishing second derivative at the origin.

Analogies clearly exist concerning concave functions.

Corollary 6.39 *Let f be twice differentiable around x_0.*

a) If x_0 is an inflection point, then $f''(x_0) = 0$.

b) Assume $f''(x_0) = 0$. If f'' changes sign when crossing x_0, then x_0 is an inflection point (ascending if $f''(x) \leq 0$ at the left of x_0 and $f''(x) \geq 0$ at its right, descending otherwise). If f'' does not change sign, x_0 is not an inflection point.

The proof relies on Taylor's formula, and will be given in Sect. 7.4.

The reader ought to beware that $f''(x_0) = 0$ does not warrant x_0 is a point of inflection for f. The function $f(x) = x^4$ has second derivative $f''(x) = 12x^2$ which vanishes at $x_0 = 0$. The origin is nonetheless not an inflection point, for the tangent at x_0 is the axis $y = 0$, and the graph of f stays always above it. In addition, f'' *does not change sign* around x_0.

Example 6.29 (continuation)

For $f(x) = xe^{2x}$ we have $f''(x) = 4(x+1)e^{2x}$ vanishing at $x_1 = -1$. As $f''(x) > 0$ if and only if $x > -1$, f is strictly concave on $(-\infty, -1)$ and strictly convex on $(-1, +\infty)$. The point $x_1 = -1$ is an ascending inflection. The graph of $f(x)$ is shown in Fig. 6.11. □

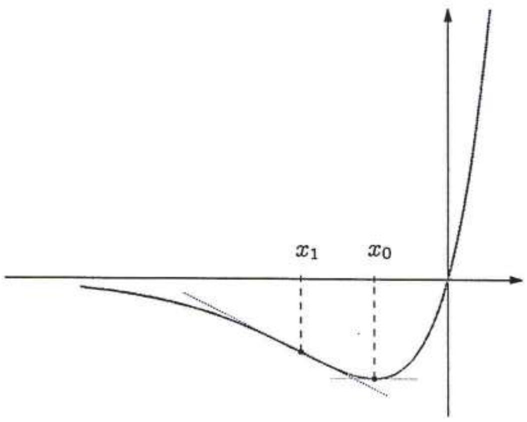


Figure 6.11. Example 6.29

6.9.1 Extension of the notion of convexity

The geometrical nature of convex maps manifests itself by considering a generalisation of the notion given in Sect. 6.9. Recall a subset C of the plane is said **convex** if the segment $\overline{P_1P_2}$ between any two points $P_1, P_2 \in C$ is all contained in C.

Given a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, we denote by

$$E_f = \{(x, y) \in \mathbb{R}^2 : x \in I,\ y \geq f(x)\}$$

the set of points of the plane lying above the graph of f (as in Fig. 6.12, left).

Definition 6.40 *The map $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called* **convex on** I *if the set E_f is a convex subset of the plane.*

It is easy to convince oneself that the convexity of E_f can be checked by considering points P_1, P_2 belonging to the graph of f only. In other words, given

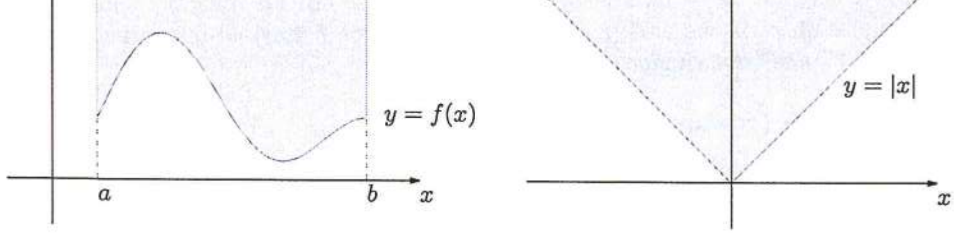


Figure 6.12. The set E_f for a generic f defined on I (left) and for $f(x) = |x|$ (right)

x_1, x_2 in I, the segment S_{12} between $(x_1, f(x_1))$ and $(x_2, f(x_2)$ should lie above the graph.

Since one can easily check that any x between x_1 and x_2 can be represented as

$$x = (1-t)x_1 + tx_2 \qquad \text{with} \quad t = \frac{x - x_1}{x_2 - x_1} \in [0,1],$$

the convexity of f reads

$$f\big((1-t)x_1 + tx_2\big) \leq (1-t)f(x_1) + tf(x_2) \qquad \forall x_1, x_2 \in I, \forall t \in [0,1].$$

If the inequality is strict for $x_1 \neq x_2$ and $t \in (0,1)$, the function is called **strictly convex on** I.

For *differentiable* functions on the interval I, Definitions 6.40, 6.33 can be proven to be equivalent. But a function may well be convex according to Definition 6.40 without being differentiable on I, like $f(x) = |x|$ on $I = \mathbb{R}$ (Fig. 6.12, right). Note, however, that convexity implies continuity at all interior points of I, although discontinuities may occur at the end-points.

6.10 Qualitative study of a function

We have hitherto supplied the reader with several analytical tools to study a map f on its domain and draw a relatively thorough – qualitatively speaking – graph. This section describes a step-by-step procedure for putting together all the information acquired.

Domain and symmetries

It should be possible to determine the domain of a generic function starting from the elementary functions that build it via algebraic operations and composition.
The study is greatly simplified if one detects the map's possible symmetries and periodicity at the very beginning (see Sect. 2.6). For instance, an even or odd map can be studied only for positive values of the variable. We point out that a function might present different kinds of symmetries, like the symmetry with respect to a vertical line other than the y-axis: the graph of $f(x) = e^{-|x-2|}$ is symmetric with respect to $x = 2$ (Fig. 6.13).

For the same reason the behaviour of a periodic function is captured by its restriction to an interval as wide as the period.

Behaviour at the end-points of the domain

Assuming the domain is a union of intervals, as often happens, one should find the one-sided limits at the end-points of each interval. Then the existence of asymptotes should be discussed, as in Sect. 5.3.
For instance, consider

$$f(x) = \frac{\log(2-x)}{\sqrt{x^2 - 2x}}.$$

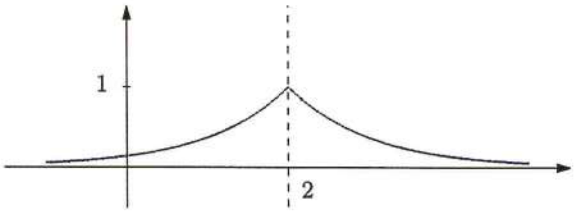


Figure 6.13. The function $f(x) = \mathrm{e}^{-|x-2|}$

Now, $\log(2-x)$ is defined for $2-x>0$, or $x<2$; in addition, $\sqrt{x^2-2x}$ has domain $x^2-2x \geq 0$, so $x \leq 0$ or $x \geq 2$, and being a denominator, $x \neq 0, 2$. Thus $\operatorname{dom} f = (-\infty, 0)$. Since $\lim_{x\to 0^-} f(x) = +\infty$, the line $x=0$ is a vertical left asymptote, while $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{\log(2-x)}{|x|} = 0$ yields the horizontal left asymptote $y=0$.

Monotonicity and extrema

The first step consists in computing the derivative f' and its domain $\operatorname{dom} f'$. Even if the derivative's analytical expression might be defined on a larger interval, one should in any case have $\operatorname{dom} f' \subseteq \operatorname{dom} f$. For example $f(x) = \log x$ has $f'(x) = \frac{1}{x}$ and $\operatorname{dom} f = \operatorname{dom} f' = (0, +\infty)$, despite $g(x) = \frac{1}{x}$ makes sense for any $x \neq 0$. After that, the zeroes and sign of f' should be determined. They allow to find the intervals where f is monotone and discuss the nature of critical points (the zeroes of f'), in the light of Sect. 6.7.

A careless analysis might result in wrong conclusions. Suppose a map f is differentiable on the union $(a,b) \cup (b,c)$ of two bordering intervals where $f' > 0$. If f is not differentiable at the point b, deducing from that that f is increasing on $(a,b) \cup (b,c)$ **is wrong**. The function $f(x) = -\frac{1}{x}$ satisfies $f'(x) = \frac{1}{x^2} > 0$ on $(-\infty, 0) \cup (0, +\infty)$, but it is not globally increasing therein (e.g. $f(-1) > f(1)$); we can only say f is increasing on $(-\infty, 0)$ and on $(0, +\infty)$ *separately.*

Recall that extremum points need not only be critical points. The function $f(x) = \sqrt{\dfrac{x}{1+x^2}}$, defined on $x \geq 0$, has a critical point $x=1$ giving an absolute maximum. At the other extremum $x=0$, the function is not differentiable, although $f(0)$ is the absolute minimum.

Convexity and inflection points

Along the same lines one determines the intervals upon which the function is convex or concave, and its inflections. As in Sect. 6.9, we use the second derivative for this.

Sign of the function and its higher derivatives

When sketching the graph of f we might find useful (not compulsory) to establish the sign of f and its vanishing points (the x-coordinates of the intersections of the graph with the horizontal axis). The roots of $f(x) = 0$ are not always easy to find analytically. In such cases one may resort to the Theorem of existence of zeroes 4.23, and deduce the presence of a unique zero within a certain interval. Likewise can be done for the sign of the first or second derivatives.

The function $f(x) = x \log x - 1$ is defined for $x > 0$. One has $f(x) < 0$ when $x \leq 1$. On $x \geq 1$ the map is strictly increasing (in fact $f'(x) = \log x + 1 > 0$ for $x > 1/\mathrm{e}$); besides, $f(1) = -1 < 0$ and $f(\mathrm{e}) = \mathrm{e} - 1 > 0$. Therefore there is exactly one zero somewhere in $(1, \mathrm{e})$, f is negative to the left of said zero and positive to the right.

6.10.1 Hyperbolic functions

An exemplary application of what seen so far is the study of a family of functions, called **hyperbolic**, that show up in various concrete situations.

We introduce the maps $f(x) = \sinh x$ and $g(x) = \cosh x$ by

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}.$$

They are respectively called **hyperbolic sine** and **hyperbolic cosine.** The terminology stems from the fundamental relation

$$\cosh^2 x - \sinh^2 x = 1\,, \qquad \forall x \in \mathbb{R}\,,$$

whence the point P of coordinates $(X, Y) = (\cosh x, \sinh x)$ runs along the right branch of the rectangular hyperbola $X^2 - Y^2 = 1$ as x varies.

The first observation is that $\operatorname{dom} f = \operatorname{dom} g = \mathbb{R}$; moreover, $f(x) = -f(-x)$ and $g(x) = g(-x)$, hence the hyperbolic sine is an odd map, whereas the hyperbolic cosine is even. Concerning the limit behaviour,

$$\lim_{x \to \pm\infty} \sinh x = \pm\infty\,, \qquad \lim_{x \to \pm\infty} \cosh x = +\infty\,.$$

This implies that there are no vertical nor horizontal asymptotes. No oblique asymptotes exist either, because these functions behave like exponentials for $x \to \infty$. More precisely

$$\sinh x \sim \pm\frac{1}{2}\mathrm{e}^{|x|}\,, \qquad \cosh x \sim \frac{1}{2}\mathrm{e}^{|x|}\,, \qquad x \to \pm\infty\,.$$

It is clear that $\sinh x = 0$ if and only if $x = 0$, $\sinh x > 0$ when $x > 0$, while $\cosh x > 0$ everywhere on $\mathbb{R}$. The monotonic features follow easily from

$$\mathrm{D} \sinh x = \cosh x \qquad \text{and} \qquad \mathrm{D} \cosh x = \sinh x\,, \qquad \forall x \in \mathbb{R}\,.$$

Thus the hyperbolic sine is increasing on the entire $\mathbb{R}$. The hyperbolic cosine is strictly increasing on $[0, +\infty)$ and strictly decreasing on $(-\infty, 0]$, has an absolute minimum $\cosh 0 = 1$ at $x = 0$ (so $\cosh x \geq 1$ on $\mathbb{R}$).

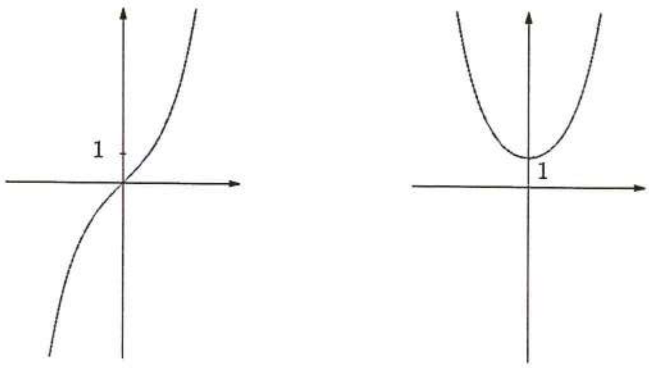


Figure 6.14. Hyperbolic sine (left) and hyperbolic cosine (right)

Differentiating once more gives

$$D^2 \sinh x = \sinh x \qquad \text{and} \qquad D^2 \cosh x = \cosh x\,, \qquad \forall x \in \mathbb{R}\,,$$

which says that the hyperbolic sine is strictly convex on $(0,+\infty)$ and strictly concave on $(-\infty,0)$. The origin is an ascending inflection point. The hyperbolic cosine is strictly convex on the whole $\mathbb{R}$. The graphs are drawn in Fig. 6.14.

In analogy to the ordinary trigonometric functions, there is a **hyperbolic tangent** defined as

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x}-1}{e^{2x}+1}\,.$$

Its domain is $\mathbb{R}$, it is odd, strictly increasing and ranges over the open interval $(-1,1)$ (Fig. 6.15).

The inverse map to the hyperbolic sine, appropriately called **inverse hyperbolic sine**, is defined on all of $\mathbb{R}$, and can be made explicit by means of the logarithm (inverse of the exponential)

$$\sinh^{-1} x = \log(x+\sqrt{x^2+1})\,, \qquad x \in \mathbb{R}\,. \tag{6.15}$$

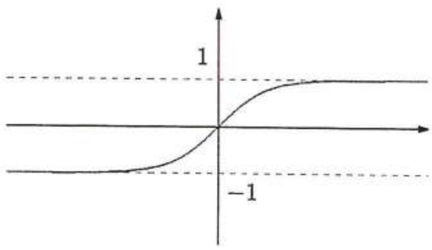


Figure 6.15. Hyperbolic tangent

There normally is no confusion with the reciprocal $1/\sinh x$, whence the use of notation[1]. The **inverse hyperbolic cosine** is obtained by inversion of the hyperbolic cosine restricted to $[0, +\infty)$

$$\boxed{\cosh^{-1} x = \log(x + \sqrt{x^2 - 1}), \qquad x \in [1, +\infty).} \tag{6.16}$$

To conclude, the **inverse hyperbolic tangent** inverts the corresponding hyperbolic map on $\mathbb{R}$

$$\boxed{\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}, \qquad x \in (-1, 1).} \tag{6.17}$$

The inverse hyperbolic functions have first derivatives

$$\begin{gathered} \mathrm{D} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}, \qquad \mathrm{D} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \\ \mathrm{D} \tanh^{-1} x = \frac{1}{1-x^2}. \end{gathered} \tag{6.18}$$

6.11 The Theorem of de l'Hôpital

This final section is entirely devoted to a single result, due to its relevance in computing the limits of indeterminate forms. Its proof can be found in Appendix A.4.2, p. 452. As always, c is one of x_0, x_0^+, x_0^-, $+\infty$, $-\infty$.

Theorem 6.41 *Let f, g be maps defined on a neighbourhood of c, except possibly at c, and such that*

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L,$$

where $L = 0, +\infty$ or $-\infty$. If f and g are differentiable around c, except possibly at c, with $g' \neq 0$, and if

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

exists (finite or not), then also

$$\lim_{x \to c} \frac{f(x)}{g(x)} \tag{6.19}$$

exists and equals the previous limit.

[1] Some authors also like the symbol Arcsinh.

Under said hypotheses the results states that

$$\boxed{\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.} \tag{6.20}$$

Examples 6.42

i) The limit

$$\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{\sin 5x}$$

gives rise to an indeterminate form of type $\frac{0}{0}$. Since numerator and denominator are differentiable functions,

$$\lim_{x\to 0}\frac{2e^{2x}+2e^{-2x}}{5\cos 5x}=\frac{4}{5}.$$

Therefore

$$\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{\sin 5x}=\frac{4}{5}.$$

ii) When the ratio $f'(x)/g'(x)$ is still an indeterminate form, supposing f and g are twice differentiable around c, except maybe at c, we can iterate the recipe of (6.20) by studying the limit of $f''(x)/g''(x)$, and so on.
Consider for instance the indeterminate form 0/0

$$\lim_{x\to 0}\frac{1+3x-\sqrt{(1+2x)^3}}{x\sin x}.$$

Differentiating numerator and denominator, we are lead to

$$\lim_{x\to 0}\frac{3-3\sqrt{1+2x}}{\sin x+x\cos x},$$

still of the form 0/0. Thus we differentiate again

$$\lim_{x\to 0}\frac{-\frac{3}{\sqrt{1+2x}}}{2\cos x-x\sin x}=-\frac{3}{2}.$$

Applying (6.20) twice allows to conclude

$$\lim_{x\to 0}\frac{1+3x-\sqrt{(1+2x)^3}}{\sin^2 x}=-\frac{3}{2}.$$

□

Remark 6.43 De l'Hôpital's Theorem is a sufficient condition only, for the existence of (6.19). Otherwise said, it might happen that the limit of the derivatives' difference quotient does not exist, whereas we have the limit of the functions' difference quotient. For example, set $f(x)=x+\sin x$ and $g(x)=2x+\cos x$. While the ratio f'/g' does not admit limit as $x\to+\infty$ (see Remark 4.19), the limit of f/g exists:

$$\lim_{x\to+\infty}\frac{x+\sin x}{2x+\cos x}=\lim_{x\to+\infty}\frac{x+o(x)}{2x+o(x)}=\frac{1}{2}.$$

□

6.11.1 Applications of de l'Hôpital's theorem

We survey some situations where the result of de l'Hôpital lends a helping hand.

Fundamental limits

By means of Theorem 6.41 we recover the important limits

$$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = +\infty, \qquad \lim_{x\to-\infty} |x|^\alpha e^x = 0, \qquad \forall \alpha \in \mathbb{R}, \tag{6.21}$$

$$\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = 0, \qquad \lim_{x\to 0+} x^\alpha \log x = 0, \qquad \forall \alpha > 0. \tag{6.22}$$

These were presented in (5.6) in the equivalent formulation of the Landau symbols. Let us begin with the first of (6.21) when $\alpha = 1$. From (6.20)

$$\lim_{x\to+\infty} \frac{e^x}{x} = \lim_{x\to+\infty} \frac{e^x}{1} = +\infty.$$

For any other $\alpha > 0$, we have

$$\lim_{x\to+\infty} \frac{e^x}{x^\alpha} = \lim_{x\to+\infty} \left(\frac{1}{\alpha} \frac{e^{\frac{x}{\alpha}}}{\frac{x}{\alpha}} \right)^\alpha = \frac{1}{\alpha^\alpha} \left(\lim_{y\to+\infty} \frac{e^y}{y} \right)^\alpha = +\infty.$$

At last, for $\alpha \le 0$ the result is rather trivial because there is no indeterminacy. As for the second formula of (6.21)

$$\lim_{x\to-\infty} |x|^\alpha e^x = \lim_{x\to-\infty} \frac{|x|^\alpha}{e^{-x}} = \lim_{x\to-\infty} \frac{|x|^\alpha}{e^{|x|}} = \lim_{y\to+\infty} \frac{y^\alpha}{e^y} = 0.$$

Now to (6.22):

$$\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \frac{1}{\alpha} \lim_{x\to+\infty} \frac{1}{x^\alpha} = 0$$

and

$$\lim_{x\to 0+} x^\alpha \log x = \lim_{x\to 0+} \frac{\log x}{x^{-\alpha}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{(-\alpha) x^{-\alpha-1}} = -\frac{1}{\alpha} \lim_{x\to 0+} x^\alpha = 0.$$

Proof of Theorem 6.15

We are now in a position to prove this earlier claim.

Proof. By definition only,

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0};$$

but this is an indeterminate form, since

$$\lim_{x\to x_0} \big(f(x) - f(x_0)\big) = \lim_{x\to x_0} (x - x_0) = 0,$$

hence de l'Hôpital implies

$$f'(x_0) = \lim_{x\to x_0} \frac{f'(x)}{1}.$$

□

Computing the order of magnitude of a map

Through examples we explain how de l'Hôpital's result detects the order of magnitude of infinitesimal or infinite functions, and their principal parts.

The function

$$f(x) = e^x - 1 - \sin x$$

is infinitesimal for $x \to 0$. With infinitesimal test function $\varphi(x) = x$ we apply the theorem twice (supposing for a moment this is possible)

$$\lim_{x\to 0} \frac{e^x - 1 - \sin x}{x^\alpha} = \lim_{x\to 0} \frac{e^x - \cos x}{\alpha x^{\alpha-1}} = \lim_{x\to 0} \frac{e^x + \sin x}{\alpha(\alpha-1)x^{\alpha-2}}.$$

When $\alpha = 2$ the right-most limit exists and is in fact $\frac{1}{2}$. This fact alone justifies the use of de l'Hôpital's Theorem. Thus $f(x)$ is infinitesimal of order 2 at the origin with respect to $\varphi(x) = x$; its principal part is $p(x) = \frac{1}{2}x^2$.

Next, consider

$$f(x) = \tan x,$$

an infinite function for $x \to \frac{\pi}{2}^-$. Setting $\varphi(x) = \dfrac{1}{\frac{\pi}{2} - x}$, we have

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\left(\frac{1}{\frac{\pi}{2}-x}\right)^\alpha} = \lim_{x\to\frac{\pi}{2}^-} \sin x \lim_{x\to\frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2}-x\right)^\alpha}{\cos x}.$$

While the first limit is 1, for the second we apply de l'Hôpital's Theorem

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2}-x\right)^\alpha}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{-\alpha(\frac{\pi}{2}-x)^{\alpha-1}}{-\sin x}.$$

The latter equals 1 when $\alpha = 1$, so $\tan x$ is infinite of first order, for $x \to \frac{\pi}{2}^-$, with respect to $\varphi(x) = \dfrac{1}{\frac{\pi}{2} - x}$. The principal part is indeed $\varphi(x)$.

6.12 Exercises

1. *Discuss differentiability at the point x_0 indicated:*

a) $f(x) = x + |x-1|$, $x_0 = 1$

[b)] $f(x) = \sin|x|$, $x_0 = 0$

[c)] $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x_0 = 0$

d) $f(x) = \sqrt{1+x^3}$, $x_0 = -1$

2. *Say where the following maps are differentiable and find the derivatives:*

a) $f(x) = x\sqrt{|x|}$ b) $f(x) = \cos|x|$

c) $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0, \\ e^x - x & \text{if } x < 0 \end{cases}$ d) $f(x) = \begin{cases} x^2+x-5 & \text{if } x \geq 1, \\ x-4 & \text{if } x < 1 \end{cases}$

3. *Compute, where defined, the first derivative of:*

a) $f(x) = 3x\sqrt[3]{1+x^2}$ b) $f(x) = \log|\sin x|$

c) $f(x) = \cos\left(e^{x^2+1}\right)$ d) $f(x) = \dfrac{1}{x\log x}$

4. *On the given interval, find maximum and minimum of:*

a) $f(x) = \sin x + \cos x\,, \qquad [0, 2\pi]$

b) $f(x) = x^2 - |x+1| - 2\,, \qquad [-2, 1]$

5. *Write the equation of the tangent at x_0 to the graph of the following maps:*

a) $f(x) = \log(3x-2)\,, \qquad x_0 = 2$ b) $f(x) = \dfrac{x}{1+x^2}\,, \qquad x_0 = 1$

c) $f(x) = e^{\sqrt{2x+1}}\,, \qquad x_0 = 0$ d) $f(x) = \sin\dfrac{1}{x}\,, \qquad x_0 = \dfrac{1}{\pi}$

6. *Verify that $f(x) = 5x + x^3 + 2x^5$ is invertible on $\mathbb{R}$, f^{-1} is differentiable on the same set, and compute $(f^{-1})'(0)$ and $(f^{-1})'(8)$.*

7. *Prove that $f(x) = (x-1)e^{x^2} + \arctan(\log x) + 2$ is invertible on its domain and find the range.*

8. *Verify that $f(x) = \log(2+x) + 2\dfrac{x+1}{x+2}$ has no zeroes apart from $x_0 = -1$.*

9. *Determine the number of zeroes and critical points of*

$$f(x) = \frac{x\log x - 1}{x^2}.$$

10. *Discuss relative and absolute minima of the map*

$$f(x) = 2\sin x + \frac{1}{2}\cos 2x$$

on $[0,\, 2\pi]$.

11. *Find the largest interval containing $x_0 = \frac{1}{2}$ on which the function*

$$f(x) = \log x - \frac{1}{\log x}$$

has an inverse, which is also explicitly required. Calculate the derivative of the inverse at the origin.

12. *Verify that*

$$\log(1 + x) \le x\,, \qquad \forall x > -1\,.$$

13. *Sketch a graph for $f(x) = 3x^5 - 50x^3 + 135x$. Then find the largest and smallest possible numbers of real roots of $f(x) + k$, as k varies in the reals.*

14. *Consider $f(x) = x^4 - 2\sqrt{\log x}$ and*

a) *find its domain;*
b) *discuss monotonicity;*
c) *prove the point $(e^4 - 2, e)$ belongs to the graph of f^{-1}, then compute the derivative of f^{-1} at $e^4 - 2$.*

15. *Regarding*

$$f(x) = \frac{\sqrt{x^2 - 3}}{x + 1},$$

a) *find domain, limits at the domain's boundary and possible asymptotes;*
b) *study the intervals of monotonicity, the maximum and minimum points, specifying which are relative, which absolute;*
c) *sketch a graph;*
d) *define*

$$g(x) = \begin{cases} f(x + \sqrt{3}) & \text{if } x \ge 0\,, \\ f(x - \sqrt{3}) & \text{if } x < 0\,. \end{cases}$$

Relying on the results found for f draw a picture of g, and study its continuity and differentiability at the origin.

16. *Given*

$$f(x) = \sqrt{|x^2 - 4|} - x,$$

a) *find domain, limits at the domain's boundary and asymptotes;*
b) *determine the sign of f;*
c) *study the intervals of monotonicity and list the extrema;*
d) *detect the points of discontinuity and of non-differentiability;*
e) *sketch the graph of f.*

17. *Consider*

$$f(x) = \sqrt[3]{e^{2x} - 1}.$$

a) *What does $f(x)$ do at the boundary of the domain?*
b) *Where is f monotone, where not differentiable?*
c) *Discuss convexity and find the inflection points.*
d) *Sketch a graph.*

18. *Let*

$$f(x) = 1 - e^{-|x|} + \frac{x}{e}$$

be given.
a) *Find domain and asymptotes, if any;*
b) *discuss differentiability and monotonic properties;*
c) *determine maxima, minima, saying whether global or local;*
d) *sketch the graph.*

19. *Given*

$$f(x) = e^x(x^2 - 8|x-3| - 8),$$

determine
a) *the monotonicity;*
b) *the relative extrema and range* $\operatorname{im} f$*;*
c) *the points where f is not continuous, or not differentiable;*
d) *a rough graph;*
e) *whether there is a real α such that*

$$g(x) = f(x) - \alpha|x-3|$$

is of class $\mathcal{C}^1$ over the whole real line.

20. *Given*

$$f(x) = \frac{\log|1+x|}{(1+x)^2},$$

find
a) *domain, behaviour at the boundary, asymptotes,*
b) *monotonicity intervals, relative or absolute maxima and minima,*
c) *convexity and inflection points,*
d) *and sketch a graph.*

21. *Let*

$$f(x) = \frac{x\log|x|}{1+\log^2|x|}.$$

a) *Prove f can be prolonged with continuity to $\mathbb{R}$ and discuss the differentiability of the prolongation g;*
b) *determine the number of stationary points g has;*
c) *draw a picture for g that takes monotonicity and asymptotes into account.*

22. *Determine for*

$$f(x) = \arctan \frac{|x|+3}{x-3}$$

a) *domain, limits at the boundary, asymptotes;*
b) *monotonicity, relative and absolute extremum points,* $\inf f$ *and* $\sup f$*;*
c) *differentiability;*
d) *concavity and convexity;*
e) *a graph that highlights the previous features.*

23. *Consider the map*

$$f(x) = \arcsin \sqrt{2e^x - e^{2x}}$$

and say
a) *what are the domain, the boundary limits, the asymptotes of* $f(x)$*;*
b) *at which points the function is differentiable;*
c) *where* f *is monotone, where it reaches a maximum or a minimum;*
d) *what the graph of* $f(x)$ *looks like, using the information so far collected.*
e) *Define a map* $\tilde{f}$ *continuously prolonging* f *to the entire* $\mathbb{R}$.

6.12.1 Solutions

1. *Differentiability:*

a) Not differentiable.
b) The right and left limits of the difference quotient, for $x \to 0$, are:

$$\lim_{x\to 0^+} \frac{\sin x - 0}{x - 0} = 1\,, \qquad \lim_{x\to 0^-} \frac{\sin(-x) - 0}{x - 0} = -1\,.$$

Consequently, the function is not differentiable at $x_0 = 0$.

c) For $x \neq 0$ the map is differentiable and

$$f'(x) = \frac{2}{x^3} e^{-1/x^2}\,.$$

Moreover $\lim_{x\to 0} f(x) = \lim_{x\to 0} f'(x) = 0$, so f is continuous at $x_0 = 0$. By Theorem 6.15, it is also differentiable at that point.

d) Not differentiable.

2. *Differentiability:*

a) Because

$$f(x) = \begin{cases} x\sqrt{x} & \text{if } x \geq 0\,,\\ x\sqrt{-x} & \text{if } x < 0\,,\end{cases}$$

f' is certainly differentiable at $x \neq 0$ with

$$f'(x) = \begin{cases} \frac{3}{2}\sqrt{x} & \text{if } x > 0, \\ \frac{3}{2}\sqrt{-x} & \text{if } x < 0. \end{cases}$$

The map is continuous on $\mathbb{R}$ (composites and products preserve continuity), hence in particular also at $x = 0$. Furthermore, $\lim_{x\to 0^+} f'(x) = \lim_{x \to 0^-} f'(x) = 0$, making f differentiable at $x = 0$, with $f'(0) = 0$.

b) Differentiable on $\mathbb{R}$, $f'(x) = -\sin x$.

c) Differentiable everywhere, $f'(x) = \begin{cases} 2x & \text{if } x \geq 0, \\ e^x - 1 & \text{if } x < 0. \end{cases}$

d) The map is clearly continuous for $x \neq 1$; but also at $x = 1$, since

$$\lim_{x\to 1^+} (x^2 + x - 5) = f(1) = -3 = \lim_{x\to 1^-} (x - 4).$$

The derivative is

$$f'(x) = \begin{cases} 2x+1 & \text{if } x > 1, \\ 1 & \text{if } x < 1, \end{cases}$$

so f is differentiable at least on $\mathbb{R} \setminus \{1\}$. Using Theorem 6.15 on the right- and left-hand derivatives independently, gives

$$f'_+(1) = \lim_{x\to 1^+} f'(x) = 3, \quad f'_-(1) = \lim_{x\to 1^-} f'(x) = 1.$$

At the point $x = 1$, a corner, the function is not differentiable.

3. *Derivatives:*

a) $f'(x) = \dfrac{5x^2+3}{(1+x^2)^{2/3}}$

b) $f'(x) = \cot\text{an}\, x$

c) $f'(x) = -2xe^{x^2+1} \sin e^{x^2+1}$

d) $f'(x) = -\dfrac{\log x + 1}{x^2 \log^2 x}$

4. *Maxima and minima:*

Both functions are continuous so the existence of maxima and minima is guaranteed by Weierstrass's theorem.

a) Maximum value $\sqrt{2}$ at the point $x = \frac{\pi}{4}$; minimum $-\sqrt{2}$ at $x = \frac{5}{4}\pi$. (The interval's end-points are relative minimum and maximum points, not absolute.)

b) One has

$$f(x) = \begin{cases} x^2 + x - 1 & \text{if } x < -1, \\ x^2 - x - 3 & \text{if } x \geq -1. \end{cases}$$

The function coincides with the parabola $y = (x + \frac{1}{2})^2 - \frac{5}{4}$ for $x < -1$. The latter has vertex in $(-\frac{1}{2}, -\frac{5}{4})$ and is convex, so on the interval $[-2, -1]$ of concern it decreases; its maximum is 1 at $x = -2$ and minimum -1 at $x = -1$.

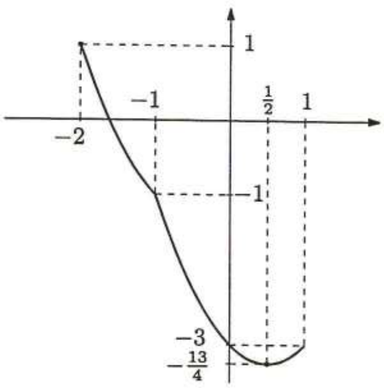


Figure 6.16. Graph of $f(x) = x^2 - |x+1| - 2$

For $x \geq -1$, we have the convex parabola $y = (x - \frac{1}{2})^2 - \frac{13}{4}$ with vertex $(\frac{1}{2}, -\frac{13}{4})$. Thus on $[-1, 1]$, there is a minimum point $x = \frac{1}{2}$ with image $f(\frac{1}{2}) = -\frac{13}{4}$. Besides, $f(-1) = -1$ and $f(1) = -3$, so the maximum -1 is reached at $x = -1$. In conclusion, f has minimum $-\frac{13}{4}$ (for $x = \frac{1}{2}$) and maximum 1 (at $x = -2$); see Fig. 6.16.

5. *Tangent lines:*

a) Since

$$f'(x) = \frac{3}{3x - 2}, \qquad f(2) = \log 4, \qquad f'(2) = \frac{3}{4},$$

the equation of the tangent is

$$y = f(2) + f'(2)(x - 2) = \log 4 + \frac{3}{4}(x - 2).$$

b) $y = \frac{1}{2}$.

c) As

$$f'(x) = \frac{e^{\sqrt{2x+1}}}{\sqrt{2x+1}}, \qquad f(0) = f'(0) = e,$$

the tangent has equation

$$y = f(0) + f'(0)x = e + ex.$$

d) $y = \pi^2(x - \frac{1}{\pi})$.

6. As sum of strictly increasing elementary functions on $\mathbb{R}$, so is our function. Therefore invertibility follows. By continuity and because $\lim_{x \to \pm\infty} f(x) = \pm\infty$, Corollary 4.30 implies $\operatorname{im} f = \mathbb{R}$. The function is differentiable on the real line, $f'(x) = 5 + 3x^2 + 10x^4 > 0$ for all $x \in \mathbb{R}$; Theorem 6.9 tells that f^{-1} is differentiable on $\mathbb{R}$. Eventually $f(0) = 0$ and $f(1) = 8$, so

$$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{5} \qquad \text{and} \qquad (f^{-1})'(8) = \frac{1}{f'(1)} = \frac{1}{18}\,.$$

7. On the domain $(0, +\infty)$ the map is strictly increasing (as sum of strictly increasing maps), hence invertible. Monotonicity follows also from the positivity of

$$f'(x) = (2x^2 - 2x + 1)\mathrm{e}^{x^2} + \frac{1}{x(1+\log^2 x)}.$$

In addition, f is continuous, so Corollary 4.30 ensures that the range is an interval bounded by $\inf f$ and $\sup f$:

$$\inf f = \lim_{x\to 0^+} f(x) = -1 - \frac{\pi}{2} + 2 = 1 - \frac{\pi}{2}, \qquad \sup f = \lim_{x\to+\infty} f(x) = +\infty\,.$$

Therefore $\operatorname{im} f = (1 - \frac{\pi}{2}, +\infty)$.

8. The map is defined only for $x > -2$, and continuous, strictly increasing on the whole domain as

$$f'(x) = \frac{1}{x+2} + \frac{2}{(x+2)^2} > 0\,, \qquad \forall x > -2\,.$$

Therefore $f(x) < f(1) = 0$ for $x < 1$ and $f(x) > f(1) = 0$ for $x > 1$.

9. The domain is $x > 0$. The zeroes solve

$$x\log x - 1 = 0 \qquad \text{i.e.} \qquad \log x = \frac{1}{x}\,.$$

If we set $h(x) = \log x$ and $g(x) = \frac{1}{x}$, then

$$h(1) = 0 < 1 = g(1) \qquad \text{and} \qquad h(\mathrm{e}) = 1 > \frac{1}{\mathrm{e}} = g(\mathrm{e})\,;$$

Corollary 4.27 says there is an $x_0 \in (1, \mathrm{e})$ such that $h(x_0) = g(x_0)$. Such a point has to be unique because h is strictly increasing and g strictly decreasing. Thus f has only one vanishing point, confined inside $(1, \mathrm{e})$.
For the critical points, we compute the first derivative:

$$f'(x) = \frac{x^2(\log x + 1) - 2x(x\log x - 1)}{x^4} = \frac{x + 2 - x\log x}{x^3}\,.$$

The zeroes of f' are then the roots of

$$x + 2 - x\log x = 0 \qquad \text{i.e.} \qquad \log x = \frac{2+x}{x}\,.$$

Let $\bar{g}(x) = \frac{2+x}{x} = 1 + \frac{2}{x}$, whence

$$h(\mathrm{e}) = 1 < 1 + \frac{2}{\mathrm{e}} = \bar{g}(\mathrm{e}) \qquad \text{and} \qquad h(\mathrm{e}^2) = 2 > 1 + \frac{2}{\mathrm{e}^2} = \bar{g}(\mathrm{e}^2)\,;$$

again, Corollary 4.27 indicates a unique $\bar{x}_0 \in (\mathrm{e}, \mathrm{e}^2)$ with $h(\bar{x}_0) = \bar{g}(\bar{x}_0)$ (uniqueness follows from the monotonicity of h and $\bar{g}$). In conclusion, f has precisely one critical point, lying in $(\mathrm{e}, \mathrm{e}^2)$.

10. In virtue of the duplication formulas (2.13),

$$f'(x) = 2\cos x - \sin 2x = 2\cos x(1 - \sin x).$$

Thus $f'(x) = 0$ when $x = \frac{\pi}{2}$ and $x = \frac{3}{2}\pi$, $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$ or $\frac{3}{2}\pi < x < 2\pi$. This says $x = \frac{\pi}{2}$ is an absolute maximum point, where $f(\frac{\pi}{2}) = \frac{3}{2}$, while $x = \frac{3}{2}\pi$ gives an absolute minimum $f(\frac{3}{2}\pi) = -\frac{5}{2}$. Additionally, $f(0) = f(2\pi) = \frac{1}{2}$ so the boundary of $[0, 2\pi]$ are extrema: more precisely, $x = 0$ is a minimum point and $x = 2\pi$ a maximum point.

11. Since f is defined on $x > 0$ with $x \neq 1$, the maximal interval containing $x_0 = \frac{1}{2}$ where f is invertible must be a subset of $(0, 1)$. On the latter, we study the monotonicity, or equivalently the invertibility, of f which is, remember, continuous everywhere on the domain. Since

$$f'(x) = \frac{1}{x} + \frac{1}{x\log^2 x} = \frac{\log^2 x + 1}{x\log^2 x},$$

it is immediate to see $f'(x) > 0$ for any $x \in (0, 1)$, meaning f is strictly increasing on $(0, 1)$. Therefore the largest interval of invertibility is indeed $(0, 1)$.
To write the inverse explicitly, put $t = \log x$ so that

$$y = t - \frac{1}{t}, \qquad t^2 - ty - 1 = 0, \qquad t = \frac{y \pm \sqrt{y^2 + 4}}{2},$$

and changing variable back to x,

$$x = \mathrm{e}^{\frac{y \pm \sqrt{y^2+4}}{2}}.$$

Being interested in $x \in (0, 1)$ only, we have

$$x = f^{-1}(y) = \mathrm{e}^{\frac{y - \sqrt{y^2+4}}{2}},$$

or, in the more customary notation,

$$y = f^{-1}(x) = \mathrm{e}^{\frac{x - \sqrt{x^2+4}}{2}}.$$

Eventually $f^{-1}(0) = \mathrm{e}^{-1}$, so

$$(f^{-1})'(0) = \frac{1}{f'(\mathrm{e}^{-1})} = \frac{1}{2\mathrm{e}}.$$

12. The function $f(x) = \log(1+x) - x$ is defined on $x > -1$, and

$$\lim_{x\to -1^+} f(x) = -\infty\,, \qquad \lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \big(-x + o(x)\big) = -\infty\,.$$

As

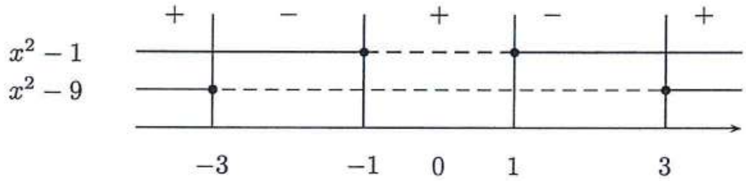
$$f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}\,,$$

$x = 0$ is critical, plus $f'(x) > 0$ on $x < 0$ and $f'(x) < 0$ for $x > 0$. Thus f increases on $(-1, 0]$ and decreases on $[0, +\infty)$; $x = 0$ is the point where the absolute maximum $f(0) = 0$ is reached. In conclusion $f(x) \le f(0) = 0$, for all $x > -1$.

13. One checks f is odd, plus

$$\begin{aligned} f'(x) &= 15x^4 - 150x^2 + 135 = 15(x^4 - 10x^2 + 9) \\ &= 15(x^2-1)(x^2-9) = 15(x+1)(x-1)(x+3)(x-3)\,. \end{aligned}$$

The sign of f' is summarised in the diagram:



What this tells is that f is increasing on $(-\infty, -3]$, $[-1, 1]$ and $[3, +\infty)$, while decreasing on $[-3, -1]$ and $[1, 3]$. The points $x = -1$, $x = 3$ are relative minima, $x = 1$ and $x = -3$ relative maxima:

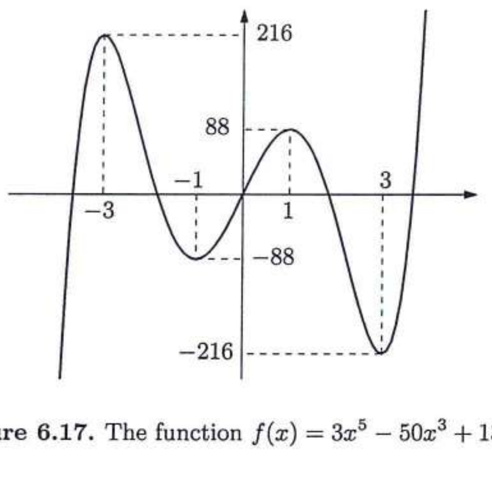


Figure 6.17. The function $f(x) = 3x^5 - 50x^3 + 135x$

$$f(1) = -f(-1) = 88 \qquad \text{and} \qquad f(3) = -f(-3) = -216.$$

Besides,

$$\lim_{x\to-\infty} f(x) = -\infty, \qquad \lim_{x\to+\infty} f(x) = +\infty.$$

The graph of f is in Fig. 6.17.

The second problem posed is equivalent to studying the number of solutions of $f(x) = -k$ as k varies: this is the number of intersections between the graph of f and the line $y = -k$. Indeed,

if $k > 216$ or $k < -216$	one solution
if $k = \pm 216$	two solutions
if $k \in (-216, -88) \cup (88, 216)$	three solutions
if $k = \pm 88$	four solutions
if $k \in (-88, 88)$	five solutions.

This gives the maximum (5) and minimum (1) number of roots of the polynomial $3x^5 - 50x^3 + 135x + k$.

14. *Study of the function* $f(x) = x^4 - 2\sqrt{\log x}$:

a) Necessarily $x > 0$ and $\log x \geq 0$, i.e., $x \geq 1$, so $\operatorname{dom} f = [1, +\infty)$.

b) From

$$f'(x) = \frac{4x^4\sqrt{\log x} - 1}{x\sqrt{\log x}}$$

we have

$$f'(x) = 0 \quad \Longleftrightarrow \quad 4x^4\sqrt{\log x} = 1 \quad \Longleftrightarrow \quad g_1(x) = \log x = \frac{1}{16x^8} = g_2(x)\,.$$

On $x \geq 1$ there is an intersection x_0 between the graphs of g_1, g_2 (Fig. 6.18). Hence $f'(x) > 0$ for $x > x_0$, f is decreasing on $[1, x_0]$, increasing on $[x_0, +\infty)$.

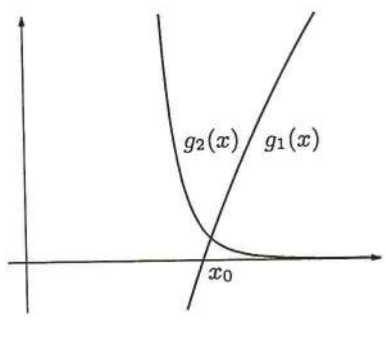


Figure 6.18. Graphs of $g_1(x) = \log x$ and $g_2(x) = \dfrac{1}{16x^8}$

This makes x_0 a minimum point, and monotonicity gives f invertible on $[1, x_0]$ and $[x_0, +\infty)$. In addition, $g_1(1) = \log 1 = 0 < \frac{1}{16} = g_2(1)$ and $g_1(2) = \log 2 > \frac{1}{2^{12}} = g_2(2)$, which implies $1 < x_0 < 2$.

c) As $f(\mathrm{e}) = \mathrm{e}^4 - 2$, the point $(\mathrm{e}^4 - 2, \mathrm{e})$ belongs to the graph of f^{-1} and

$$(f^{-1})'(\mathrm{e}^4 - 2) = \frac{1}{f'(\mathrm{e})} = \frac{\mathrm{e}}{4\mathrm{e}^4 - 1}.$$

15. *Study of* $f(x) = \frac{\sqrt{x^2-3}}{x+1}$:

a) The domain is determined by $x^2 - 3 \geq 0$ together with $x \neq -1$, hence $\operatorname{dom} f = (-\infty, -\sqrt{3}] \cup [\sqrt{3}, +\infty)$. At the boundary points:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{|x|\sqrt{1 - \frac{3}{x^2}}}{x(1 + \frac{1}{x})} = \lim_{x\to\pm\infty} \frac{|x|}{x} = \pm 1\,,$$
$$\lim_{x\to-\sqrt{3}^-} f(x) = \lim_{x\to\sqrt{3}^+} f(x) = 0,$$

so $y = 1$ is the horizontal right asymptote, $y = -1$ the horizontal left asymptote.

b) The derivative

$$f'(x) = \frac{x+3}{(x+1)^2\sqrt{x^2-3}}$$

vanishes at $x = -3$ and is positive for $x \in (-3, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$. Thus f is increasing on $[-3, -\sqrt{3}]$ and $[\sqrt{3}, +\infty)$, decreasing on $(-\infty, -3]$; $x = -3$ is an absolute minimum with $f(-3) = -\frac{\sqrt{6}}{2} < -1$. Furthermore, the points $x = \pm\sqrt{3}$ are extrema too, namely $x = -\sqrt{3}$ is a relative maximum, $x = \sqrt{3}$ a relative minimum: $f(\pm\sqrt{3}) = 0$.

c) Fig. 6.19 (left) shows the graph of f.

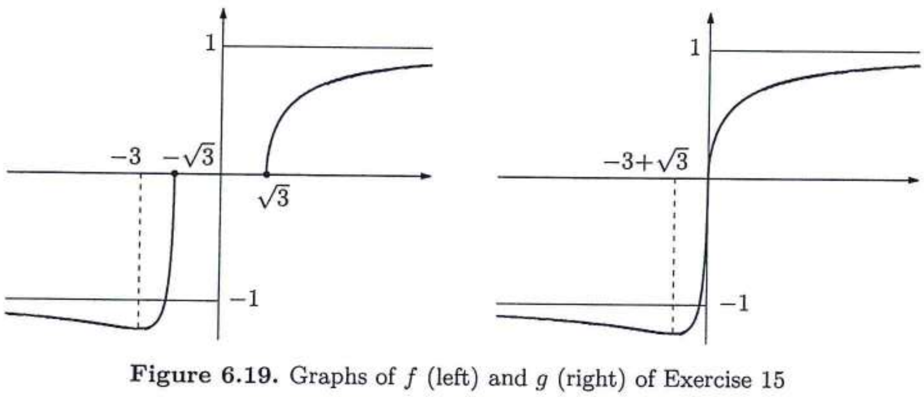


Figure 6.19. Graphs of f (left) and g (right) of Exercise 15

d) Right-translating the negative branch of f by $\sqrt{3}$ gives $g(x)$ for $x < 0$, whereas shifting to the left the branch on $x > 0$ gives the positive part of g. The final result is shown in Fig. 6.19 (right).
The map g is continuous on $\mathbb{R}$, in particular

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} f(x-\sqrt{3}) = f(-\sqrt{3}) = 0 = f(\sqrt{3}) = \lim_{x\to 0+} g(x).$$

Since

$$\lim_{x\to 0^\pm} g'(x) = \lim_{x\to\sqrt{3}^+} f'(x) = \lim_{x\to -\sqrt{3}^-} f'(x) = +\infty$$

g is not differentiable at $x = 0$.

16. *Study of* $f(x) = \sqrt{|x^2-4|} - x$:

a) The domain is $\mathbb{R}$ and

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2-4-x^2}{\sqrt{x^2-4}+x} = 0^-, \qquad \lim_{x\to-\infty} f(x) = +\infty.$$

Thus $y = 0$ is a horizontal right asymptote. Let us search for oblique asymptotic directions. As

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \left(-\sqrt{1-\frac{4}{x^2}} - 1\right) = -2\,,$$

$$\lim_{x\to-\infty} (f(x)+2x) = \lim_{x\to-\infty} \left(\sqrt{x^2-4}+x\right) = \lim_{x\to-\infty} \frac{x^2-4-x^2}{\sqrt{x^2-4}-x} = 0,$$

the line $y = -2x$ is an oblique left asymptote.

b) It suffices to solve $\sqrt{|x^2-4|} - x \geq 0$. First, $\sqrt{|x^2-4|} \geq x$ for any $x < 0$. When $x \geq 0$, we distinguish two cases: $x^2-4 < 0$ (so $0 \leq x < 2$) and $x^2 - 4 \geq 0$ (i.e., $x \geq 2$).
On $0 \leq x < 2$, squaring gives

$$4 - x^2 \geq x^2 \qquad \Longleftrightarrow \qquad x^2 - 2 \leq 0 \qquad \Longleftrightarrow \qquad 0 \leq x \leq \sqrt{2}\,.$$

For $x \geq 2$, squaring implies $x^2 - 4 \geq x^2$, which holds nowhere. The function then vanishes only at $x = \sqrt{2}$, is positive on $x < \sqrt{2}$ and strictly negative for $x > \sqrt{2}$.

c) Since

$$f(x) = \begin{cases} \sqrt{4-x^2} - x & \text{if } -2 < x < 2\,, \\ \sqrt{x^2-4} - x & \text{if } x \leq -2\,,\ x \geq 2\,, \end{cases}$$

we have

$$f'(x) = \begin{cases} \dfrac{-x}{\sqrt{4-x^2}} - 1 & \text{if } -2 < x < 2\,, \\ \dfrac{x}{\sqrt{x^2-4}} - 1 & \text{if } x < -2\,,\ x > 2\,. \end{cases}$$

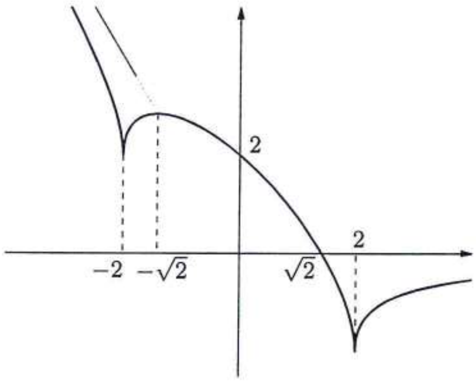


Figure 6.20. The function $f(x) = \sqrt{|x^2 - 4|} - x$

When $-2 < x < 2$, $f'(x) \geq 0$ if $x + \sqrt{4 - x^2} \leq 0$, that is $\sqrt{4 - x^2} \leq -x$. The inequality does not hold for $x \geq 0$; on $-2 < x < 0$ we square, so that

$$4 - x^2 \leq x^2 \qquad \Longleftrightarrow \qquad x^2 - 2 \geq 0 \qquad \Longleftrightarrow \qquad -2 \leq x \leq -\sqrt{2}\,.$$

Hence $f'(x) = 0$ for $x = -\sqrt{2}$, $f'(x) > 0$ for $-2 < x < -\sqrt{2}$ and $f'(x) < 0$ when $-\sqrt{2} < x < 2$.
If $x < -2$ or $x > 2$, $f'(x) \geq 0$ if $x - \sqrt{x^2 - 4} \geq 0$, i.e., $\sqrt{x^2 - 4} \leq x$. The latter is never true for $x < -2$; for $x > 2$, $x^2 \geq x^2 - 4$ is a always true. Therefore $f'(x) > 0$ per $x > 2$ e $f'(x) < 0$ per $x < -2$.
Summary: f decreases on $(-\infty, -2]$ and $[-\sqrt{2}, 2]$, increases on $[-2, -\sqrt{2}]$ and $[2, +\infty)$. The points $x = \pm 2$ are relative minima, $x = -\sqrt{2}$ a relative maximum. The corresponding values are $f(-2) = 2$, $f(2) = -2$, $f(-\sqrt{2}) = 2\sqrt{2}$, so $x = 2$ is actually a global minimum.

d) As composite of continuous elementary maps, f is continuous on its domain. To study differentiability it is enough to examine f' for $x \to \pm 2$. Because

$$\lim_{x \to \pm 2} f'(x) = \infty,$$

at $x = \pm 2$ there is no differentiability.

e) The graph is shown in Fig. 6.20.

17. *Study of* $f(x) = \sqrt[3]{e^{2x} - 1}$:

a) The function is defined everywhere

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = -1.$$

b) The first derivative

$$f'(x) = \frac{2}{3} \frac{e^{2x}}{(e^{2x} - 1)^{2/3}}$$

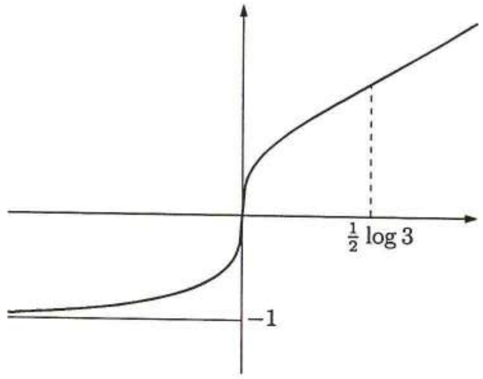


Figure 6.21. The map $f(x) = \sqrt[3]{e^{2x} - 1}$

is positive for $x \in \mathbb{R} \setminus \{0\}$, and f is not differentiable at $x = 0$, for $\lim_{x \to 0} f'(x) = +\infty$. Therefore f increases everywhere on $\mathbb{R}$.

c) The second derivative (for $x \neq 0$)

$$f''(x) = \frac{4}{9} e^{2x} \frac{e^{2x} - 3}{(e^{2x} - 1)^{5/3}}$$

vanishes at $x = \frac{1}{2}\log 3$; it is positive when $x \in (-\infty, 0) \cup (\frac{1}{2}\log 3, +\infty)$. This makes $x = \frac{1}{2}\log 3$ an ascending inflection, plus f convex on $(-\infty, 0]$ and $[\frac{1}{2}\log 3, +\infty)$, concave on $[0, \frac{1}{2}\log 3]$. Suitably extending the definition, the point $x = 0$ may be acknowledged as an inflection (with vertical tangent).

d) See Fig. 6.21.

18. *Study of* $f(x) = 1 - e^{-|x|} + \frac{x}{e}$:

a) Clearly $\operatorname{dom} f = \mathbb{R}$. As

$$\lim_{x \to \pm\infty} e^{-|x|} = 0,$$

we immediately obtain

$$\lim_{x \to \pm\infty} f(x) = \pm\infty,$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \left(\frac{1}{x} - \frac{e^{-|x|}}{x} + \frac{1}{e} \right) = \frac{1}{e},$$

$$\lim_{x \to \pm\infty} \left(f(x) - \frac{x}{e} \right) = \lim_{x \to \pm\infty} (1 - e^{-|x|}) = 1.$$

This makes $y = \frac{1}{e}x + 1$ a complete oblique asymptote.

b) The map is continuous on $\mathbb{R}$, and certainly differentiable for $x \neq 0$. As

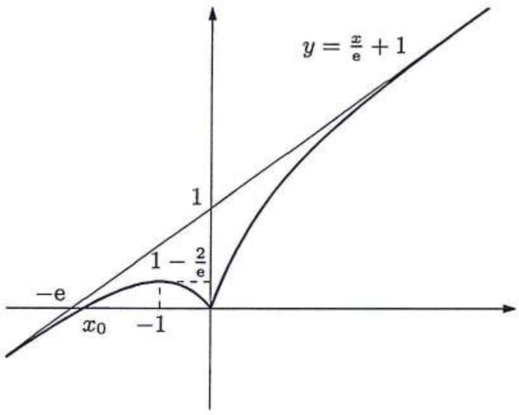


Figure 6.22. Graph of $f(x) = 1 - e^{-|x|} + \frac{x}{e}$

$$f'(x) = \begin{cases} e^{-x} + \dfrac{1}{e} & \text{if } x > 0\,, \\ -e^{x} + \dfrac{1}{e} & \text{if } x < 0\,, \end{cases}$$

it follows

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \left(-e^{x} + \frac{1}{e}\right) = \frac{1}{e} - 1$$
$$\neq \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \left(e^{-x} + \frac{1}{e}\right) = \frac{1}{e} + 1,$$

preventing differentiability at $x = 0$.
Moreover, for $x > 0$ we have $f'(x) > 0$. On $x < 0$, $f'(x) > 0$ if $e^x < \frac{1}{e}$, i.e., $x < -1$. The map is increasing on $(-\infty, -1]$ and $[0, +\infty)$, decreasing on $[-1, 0]$.

c) The previous considerations imply $x = -1$ is a local maximum with $f(-1) = 1 - \frac{2}{e}$, $x = 0$ a local minimum where $f(0) = 0$.

d) See Fig. 6.22.

19. *Study of* $f(x) = e^x(x^2 - 8|x-3| - 8)$:

a) The domain covers $\mathbb{R}$. Since

$$f(x) = \begin{cases} e^x(x^2 + 8x - 32) & \text{if } x < 3\,, \\ e^x(x^2 - 8x + 16) & \text{if } x \geq 3\,, \end{cases}$$

we have

$$f'(x) = \begin{cases} e^x(x^2 + 10x - 24) & \text{if } x < 3\,, \\ e^x(x^2 - 6x + 8) & \text{if } x > 3\,. \end{cases}$$

On $x < 3$: $f'(x) = 0$ if $x^2 + 10x - 24 = 0$, so $x = -12$ or $x = 2$, while $f'(x) > 0$ if $x \in (-\infty, -12) \cup (2, 3)$. On $x > 3$: $f'(x) = 0$ if $x^2 - 6x + 8 = 0$, i.e., $x = 4$ ($x = 2$ is a root, but lying outside the interval $x > 3$ we are considering), while $f'(x) > 0$ if $x \in (4, +\infty)$.
Therefore f is increasing on the intervals $(-\infty, -12]$, $[2, 3]$ and $[4, +\infty)$, decreasing on $[-12, 2]$ and $[3, 4]$.

b) From part a) we know $x = -12$ and $x = 3$ are relative maxima, $x = 2$ and $x = 4$ relative minima: $f(-12) = 16e^{-12}$, $f(2) = -12e^2$, $f(3) = e^3$ and $f(4) = 0$. For the range, let us determine

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^x(x^2 + 8x - 32) = 0\,,$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^x(x^2 - 8x + 16) = +\infty.$$

Continuity implies

$$\text{im } f = [\min f(x), \sup f(x)) = [f(2), +\infty) = [-12e^2, +\infty).$$

c) No discontinuities are present, for the map is the composite of continuous functions. As for the differentiability, the only unclear point is $x = 3$. But

$$\lim_{x\to3^-} f'(x) = \lim_{x\to3^-} e^x(x^2 + 10x - 24) = 15e^3\,,$$
$$\lim_{x\to3+} f'(x) = \lim_{x\to3+} e^x(x^2 - 6x + 8) = -e^3,$$

so$f$ is not differentiable at $x = 3$.

d) See Fig. 6.23; a neighbourhood of $x = -12$ is magnified.

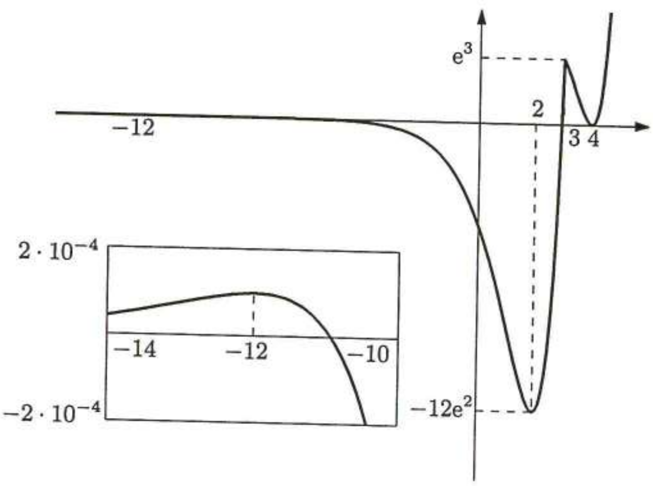


Figure 6.23. Graph of $f(x) = e^x(x^2 - 8|x - 3| - 8)$

e) The function g is continuous on the real axis and

$$g'(x) = \begin{cases} e^x(x^2+10x-24)+\alpha & \text{if } x<3\,, \\ e^x(x^2-6x+8)-\alpha & \text{if } x>3\,. \end{cases}$$

In order for g to be differentiable at $x=3$, we must have

$$\lim_{x\to 3^-} g'(x) = 15e^3+\alpha = \lim_{x\to 3^+} g'(x) = -e^3-\alpha\,;$$

the value $\alpha = -8e^3$ makes g of class $\mathcal{C}^1$ on the whole real line.

20. *Study of* $f(x) = \frac{\log|1+x|}{(1+x)^2}$:

a) $\operatorname{dom} f = \mathbb{R}\setminus\{-1\}$. By (5.6) c)

$$\lim_{x\to\pm\infty} f(x) = 0^+$$

while

$$\lim_{x\to -1^\pm} f(x) = \frac{\infty}{0^+} = -\infty\,.$$

From this, $x=-1$ is a vertical asymptote, and $y=0$ is a complete oblique asymptote.

b) The derivative

$$f'(x) = \frac{1-2\log|x+1|}{(x+1)^3}$$

tells that $f(x)$ is differentiable on the domain; $f'(x)=0$ if $|x+1|=\sqrt{e}$, hence for $x=-1\pm\sqrt{e}$; $f'(x)>0$ if $x\in(-\infty,-\sqrt{e}-1)\cup(-1,\sqrt{e}-1)$. All this says f increases on $(-\infty,-\sqrt{e}-1]$ and $(-1,-1+\sqrt{e}]$, decreases on $[-\sqrt{e}-1,-1)$ and $[-1+\sqrt{e},+\infty)$, has (absolute) maxima at $x=-1\pm\sqrt{e}$, with $f(-1\pm\sqrt{e})=\frac{1}{2e}$.

c) From

$$f''(x) = \frac{-5+6\log|x+1|}{(x+1)^4}$$

the second derivative is defined at each point of $\operatorname{dom} f$, and vanishes at $|x+1| = e^{5/6}$, so $x=-1\pm e^{5/6}$. Since $f''(x)>0$ on $x\in(-\infty,-1-e^{5/6})\cup(e^{5/6}-1,+\infty)$, f is convex on $(-\infty,-1-e^{5/6}]$ and $[e^{5/6}-1,+\infty)$, while is concave on $[-1-e^{5/6},-1)$ and $(-1,e^{5/6}-1]$. The points $x=-1\pm e^{5/6}$ are inflections.

d) See Fig. 6.24.

21. *Study of* $f(x) = \frac{x\log|x|}{1+\log^2|x|}$:

a) The domain is clear: $\operatorname{dom} f = \mathbb{R}\setminus\{0\}$. Since $\lim\limits_{x\to 0} f(x)=0$ (x 'wins' against the logarithm) we can extend f to $\mathbb{R}$ with continuity, by defining $g(0)=0$. The function is odd, so we shall restrict the study to $x>0$.

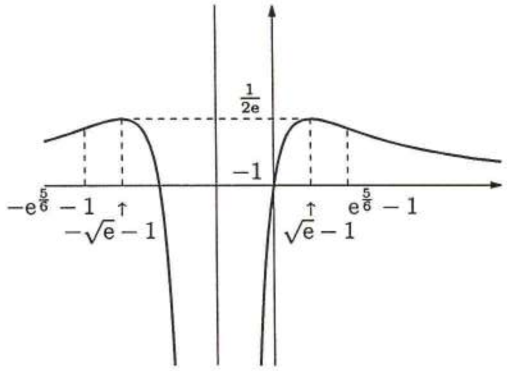


Figure 6.24. Graph of $f(x) = \frac{\log|1+x|}{(1+x)^2}$

As far as the differentiability is concerned, when $x > 0$

$$f'(x) = \frac{\log^3 x - \log^2 x + \log x + 1}{(1 + \log^2 x)^2};$$

with $t = \log x$, the limit reads

$$\lim_{x \to 0} f'(x) = \lim_{t \to -\infty} \frac{t^3 - t^2 + t + 1}{(1 + t^2)^2} = \lim_{t \to -\infty} \frac{t^3}{t^4} = 0.$$

Therefore the map g, prolongation of f, is not only continuous but also differentiable, due to Theorem 6.15, on the entire $\mathbb{R}$. In particular $g'(0) = 0$.

b) Part a) is also telling that $x = 0$ is stationary for g. To find other critical points, we look at the zeroes of the map $h(t) = t^3 - t^2 + t + 1$, where $t = \log x$ ($x > 0$). Since

$$\lim_{t \to -\infty} h(t) = -\infty, \qquad \lim_{t \to \infty} h(t) = +\infty,$$
$$h(0) = 1, \qquad h'(t) = 3t^2 - 2t + 1 > 0, \quad \forall t \in \mathbb{R},$$

h is always increasing and has one negative zero t_0. Its graph is represented in Fig. 6.25 (left).
As $t_0 = \log x_0 < 0$, $0 < x_0 = e^{t_0} < 1$. But the function is odd, so g has two more stationary points, x_0 and $-x_0$ respectively.

c) By the previous part $g'(x) > 0$ on $(x_0, +\infty)$ and $g'(x) < 0$ on $(0, x_0)$. To summarise then, g (odd) is increasing on $(-\infty, -x_0]$ and $[x_0, +\infty)$, decreasing on $[-x_0, x_0]$. Because

$$\lim_{x \to +\infty} g(x) = +\infty$$

and

$$\lim_{x \to +\infty} \frac{g(x)}{x} = \lim_{x \to +\infty} \frac{\log x}{1 + \log^2 x} = \lim_{t \to +\infty} \frac{t}{1 + t^2} = 0,$$

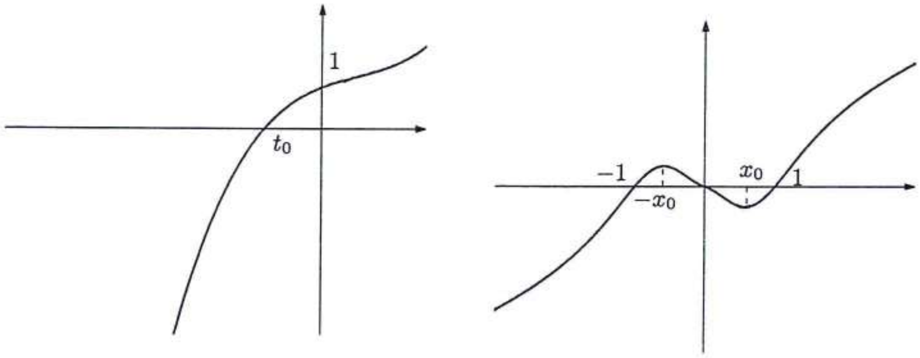


Figure 6.25. The functions h (left) and g (right) of Exercise 21

there are no asymptotes.
For the graph see Fig. 6.25 (right).

22. *Study of* $f(x) = \arctan \frac{|x|+3}{x-3}$*:*

a) $\operatorname{dom} f = \mathbb{R} \setminus \{3\}$. The function is more explicitly given by

$$f(x) = \begin{cases} \arctan \dfrac{-x+3}{x-3} = \arctan(-1) = -\dfrac{\pi}{4} & \text{if } x \leq 0\,, \\ \arctan \dfrac{x+3}{x-3} & \text{if } x > 0, \end{cases}$$

whence

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -\frac{\pi}{4} = -\frac{\pi}{4}, \qquad \lim_{x\to+\infty} f(x) = \arctan 1 = \frac{\pi}{4},$$
$$\lim_{x\to3^-} f(x) = \arctan \frac{6}{0^-} = \arctan(-\infty) = -\frac{\pi}{2},$$
$$\lim_{x\to3^+} f(x) = \arctan \frac{6}{0^+} = \arctan(+\infty) = \frac{\pi}{2}.$$

Then the straight lines $y = -\frac{\pi}{4}$, $y = \frac{\pi}{4}$ are horizontal asymptotes (left and right respectively).

b) The map

$$f'(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ -\dfrac{3}{x^2+9} & \text{if } x > 0, \quad x \neq 3, \end{cases}$$

is negative on $x > 0$, $x \neq 3$, so f is strictly decreasing on $[0,3)$ and $(3,+\infty)$, but only non-increasing on $(-\infty, 3)$. The reader should take care that f is not strictly decreasing on the whole $[0,3) \cup (3,+\infty)$ (recall the remarks of p. 197). The interval $(-\infty,0)$ consists of points of relative non-strict maxima and minima, for $f(x) = -\frac{\pi}{4}$, whereas $x = 0$ is a relative maximum.

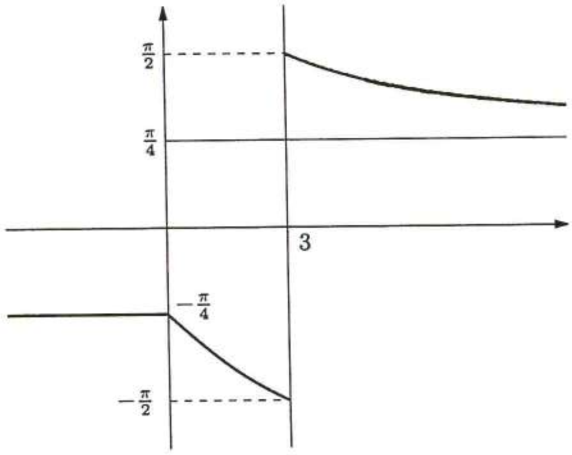


Figure 6.26. The function $f(x) = \arctan \dfrac{|x|+3}{x-3}$

Eventually, $\inf f(x) = -\frac{\pi}{2}$, $\sup f(x) = \frac{\pi}{2}$ (the map admits no maximum, nor minimum).

c) Our map is certainly differentiable on $\mathbb{R} \setminus \{0,3\}$. At $x = 3$, f is not defined; at $x = 0$, f is continuous but

$$\lim_{x \to 0^-} f'(x) = 0 \neq \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} -\frac{3}{x^2+9} = -\frac{1}{3},$$

showing that differentiability does not extend beyond $\mathbb{R} \setminus \{0,3\}$.

d) Computing

$$f''(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ \dfrac{6x}{(x^2+9)^2} & \text{if } x > 0, \quad x \neq 3\,, \end{cases}$$

reveals that $f''(x) > 0$ for $x > 0$ with $x \neq 3$, so f is convex on $[0,3)$ and $(3,+\infty)$.

e) See Fig. 6.26.

23. *Study of* $f(x) = \arcsin \sqrt{2e^x - e^{2x}}$:

a) We have to impose $2e^x - e^{2x} \geq 0$ and $-1 \leq \sqrt{2e^x - e^{2x}} \leq 1$ for the domain; the first constraint is equivalent to $2 - e^x \geq 0$, hence $x \leq \log 2$. Having assumed that square roots are always positive, the second inequality reduces to $2e^x - e^{2x} \leq 1$. With $y = e^x$, we can write $y^2 - 2y + 1 = (y-1)^2 \geq 0$, which is always true. Thus $\operatorname{dom} f = (-\infty, \log 2]$. Moreover,

$$\lim_{x \to -\infty} f(x) = 0\,, \qquad f(\log 2) = 0,$$

and $y = 0$ is a horizontal left asymptote.