b) From

$$f'(x) = \frac{e^x(1-e^x)}{\sqrt{e^x(2-e^x)(1-2e^x+e^{2x})}} = \frac{e^x(1-e^x)}{\sqrt{e^x(2-e^x)(1-e^x)^2}}$$

$$= \begin{cases} -\dfrac{e^x}{\sqrt{e^x(2-e^x)}} & \text{if } 0 < x < \log 2\,, \\ \dfrac{e^x}{\sqrt{e^x(2-e^x)}} & \text{if } x < 0\,, \end{cases}$$

we see that

$$\lim_{x\to(\log 2)^-} f'(x) = -\infty\,, \qquad \lim_{x\to 0^+} f'(x) = -1\,, \qquad \lim_{x\to 0^-} f'(x) = 1\,.$$

In this way f is not differentiable at $x = \log 2$, where the tangent is vertical, and at the corner point $x = 0$.

c) The sign of f' is positive for $x < 0$ and negative for $0 < x < \log 2$, meaning that $x = 0$ is a global maximum point, $f(0) = \frac{\pi}{2}$, while at $x = \log 2$ the absolute minimum $f(\log 2) = 0$ is reached; f is monotone on $(-\infty, 0]$ (increasing) and $[0, \log 2]$ (decreasing).

d) See Fig. 6.27.

e) A possible choice to extend f with continuity is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \le \log 2\,, \\ 0 & \text{if } x > \log 2\,. \end{cases}$$

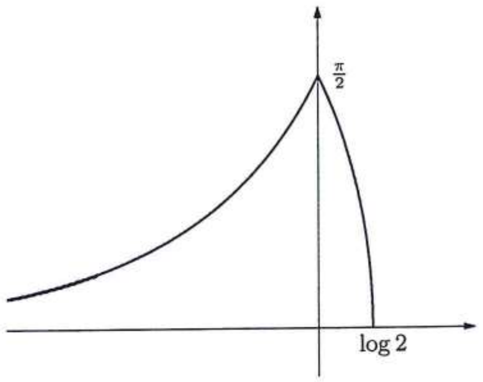

Figure 6.27. The map $f(x) = \arcsin\sqrt{2e^x - e^{2x}}$

7

Taylor expansions and applications

The Taylor expansion of a function around a real point x_0 is the representation of the map as sum of a polynomial of a certain degree and an infinitesimal function of order bigger than the degree. It provides an extremely effective tool both from the qualitative and the quantitative point of view. In a small enough neighbourhood of x_0 one can approximate the function, however complicated, using the polynomial; the qualitative features of the latter are immediate, and polynomials are easy to handle. The expansions of the main elementary functions can be aptly combined to produce more involved expressions, in a way not dissimilar to the algebra of polynomials.

7.1 Taylor formulas

We wish to tackle the problem of approximating a function f, around a given point $x_0 \in \mathbb{R}$, by polynomials of increasingly higher degree.

We begin by assuming f be continuous at x_0. Introducing the constant polynomial (degree zero)

$$Tf_{0,x_0}(x) = f(x_0), \qquad \forall x \in \mathbb{R},$$

formula (5.4) prompts us to write

$$\boxed{f(x) = Tf_{0,x_0}(x) + o(1), \quad x \to x_0.} \tag{7.1}$$

Put in different terms, we may approximate f around x_0 using a zero-degree-polynomial, in such a way that the difference $f(x) - Tf_{0,x_0}(x)$ (called *error of approximation*, or *remainder*), is infinitesimal at x_0 (Fig. 7.1). The above relation is the first instance of a Taylor formula.

Suppose now f is not only continuous but also differentiable at x_0: then the first formula of the finite increment (6.11) holds. By defining the polynomial in x of degree one

C. Canuto, A. Tabacco: *Mathematical Analysis I*, 2nd Ed.,
UNITEXT – La Matematica per il 3+2 84, DOI 10.1007/978-3-319-12772-9_7,

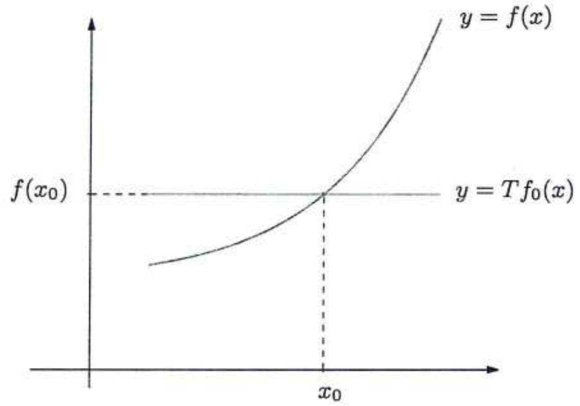

Figure 7.1. Local approximation of f by the polynomial $Tf_0 = Tf_{0,x_0}$

$$Tf_{1,x_0}(x) = f(x_0) + f'(x_0)(x - x_0),$$

whose graph is the tangent line to f at x_0 (Fig. 7.2), relation (6.11) reads

$$f(x) = Tf_{1,x_0}(x) + o(x - x_0), \quad x \to x_0. \tag{7.2}$$

This is another Taylor formula: it says that a differentiable map at x_0 can be locally approximated by a linear function, with an error of approximation that not only tends to 0 as $x \to x_0$, but is infinitesimal of order bigger than one.

In case f is differentiable in a neighbourhood of x_0, except perhaps at x_0, the second formula of the finite increment (6.13) is available: putting $x_1 = x_0$, $x_2 = x$ we write the latter as

$$f(x) = Tf_{0,x_0}(x) + f'(\bar{x})(x - x_0), \tag{7.3}$$

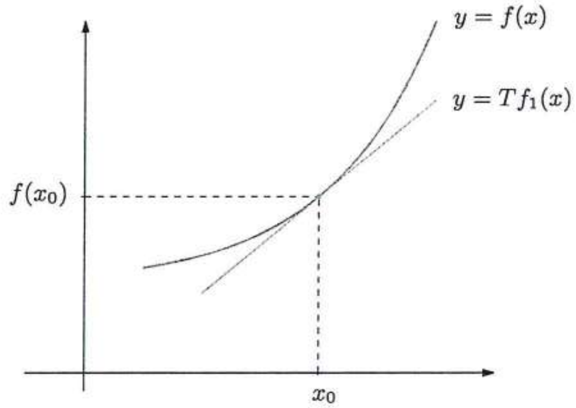

Figure 7.2. Local approximation of f by the polynomial $Tf_1 = Tf_{1,x_0}$

where $\bar{x}$ denotes a suitable point between x_0 and x. Compare this with (7.1): now we have a more accurate expression for the remainder. This allows to appraise numerically the accuracy of the approximation, once the increment $x - x_0$ and an estimate of f' around x_0 are known. Formula (7.3) is of Taylor type as well, and the remainder is called *Lagrange's remainder*. In (7.1), (7.2) we call it *Peano's remainder*, instead.

Now that we have approximated f with polynomials of degrees 0 or 1, as $x \to x_0$, and made errors $o(1) = o\big((x-x_0)^0\big)$ or $o(x-x_0)$ respectively, the natural question is whether it is possible to approximate the function by a quadratic polynomial, with an error $o\big((x-x_0)^2\big)$ as $x \to x_0$. Equivalently, we seek for a real number a such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + a(x-x_0)^2 + o\big((x-x_0)^2\big), \quad x \to x_0 . \tag{7.4}$$

This means

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - a(x-x_0)^2}{(x-x_0)^2} = 0.$$

By de l'Hôpital's Theorem, such limit holds if

$$\lim_{x \to x_0} \frac{f'(x) - f'(x_0) - 2a(x-x_0)}{2(x-x_0)} = 0,$$

i.e.,

$$\lim_{x \to x_0} \left(\frac{1}{2} \frac{f'(x) - f'(x_0)}{x - x_0} - a \right) = 0,$$

or

$$\frac{1}{2} \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = a.$$

We conclude that (7.4) is valid when the right-hand-side limit exists and is finite: in other words, when f is twice differentiable at x_0. If so, the coefficient a is $\frac{1}{2}f''(x_0)$. In this way we have obtained the Taylor formula (with Peano's remainder)

$$\boxed{f(x) = Tf_{2,x_0}(x) + o\big((x-x_0)^2\big), \quad x \to x_0,} \tag{7.5}$$

where

$$Tf_{2,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2$$

is the Taylor polynomial of f at x_0 with degree 2 (Fig. 7.3).

The recipe just described can be iterated, and leads to polynomial approximations of increasing order. The final result is the content of the next theorem.

This remainder is said **Lagrange's remainder of order** n, and (7.8) is the Taylor expansion of f at x_0 of order n with Lagrange's remainder.

Theorems 7.1 and 7.2 are proven in Appendix A.4.4, p. 456.

An additional form of the remainder of order n in a Taylor formula, called **integral remainder**, will be provided in Theorem 9.44.

A Taylor expansion centred at the origin ($x_0 = 0$) is sometimes called **Maclaurin expansion**. A useful relation to simplify the computation of a Maclaurin expansion goes as follows.

Property 7.3 *The Maclaurin polynomial of an even (respectively, odd) map involves only even (odd) powers of the independent variable.*

Proof. If f is even and n times differentiable around the origin, the claim follows from (7.7) with $x_0 = 0$, provided we show all derivatives of odd order vanish at the origin.

Recalling Property 6.12, f even implies f' odd, f'' even, f''' odd et cetera. In general, even-order derivatives $f^{(2k)}$ are even functions, whereas $f^{(2k+1)}$ are odd. But an odd map g must necessarily vanish at the origin (if defined there), because $x = 0$ in $g(-x) = -g(x)$ gives $g(0) = -g(0)$, whence $g(0) = 0$.

The argument is the same for f odd. □

7.2 Expanding the elementary functions

The general results permit to expand simple elementary functions. Other functions will be discussed in Sect. 7.3.

The exponential function

Let $f(x) = \mathrm{e}^x$. Since all derivatives are identical with e^x, we have $f^{(k)}(0) = 1$ for any $k \geq 0$. Maclaurin's expansion of order n with Peano's remainder is

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n) = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)\,. \tag{7.9}$$

Using Lagrange's remainder, we have

$$\mathrm{e}^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{\mathrm{e}^{\bar{x}}}{(n+1)!} x^{n+1}, \qquad \text{for a certain } \bar{x} \text{ between } 0 \text{ and } x. \tag{7.10}$$

Maclaurin's polynomials for e^x of order $n = 1, 2, 3, 4$ are shown in Fig. 7.4.

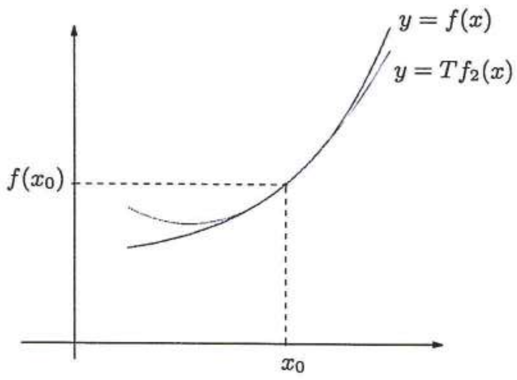

Figure 7.3. Local approximation of f by $Tf_2 = Tf_{2,x_0}$

Theorem 7.1 (Taylor formula with Peano's remainder) *Let $n \geq 0$ and f be n times differentiable at x_0. Then the* **Taylor formula** *holds*

$$f(x) = Tf_{n,x_0}(x) + o\big((x-x_0)^n\big), \quad x \to x_0, \tag{7.6}$$

where

$$\begin{aligned} Tf_{n,x_0}(x) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k \\ &= f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n. \end{aligned} \tag{7.7}$$

The term $Tf_{n,x_0}(x)$ is the **Taylor polynomial of f at x_0 of order** (or **degree**) n, while $o\big((x-x_0)^n\big)$ as in (7.6) is **Peano's remainder of order** n. The representation of f given by (7.6) is called **Taylor expansion** of f at x_0 of order n, with remainder in Peano's form.

Under stronger hypotheses on f we may furnish a preciser formula for the remainder, thus extending (7.3).

Theorem 7.2 (Taylor formula with Lagrange's remainder) *Let $n \geq 0$ and f differentiable n times at x_0, with continuous nth derivative, be given; suppose f is differentiable $n+1$ times around x_0, except possibly at x_0. Then the Taylor formula*

$$f(x) = Tf_{n,x_0}(x) + \frac{1}{(n+1)!} f^{(n+1)}(\bar{x})(x-x_0)^{n+1}, \tag{7.8}$$

holds, for a suitable $\bar{x}$ between x_0 and x.

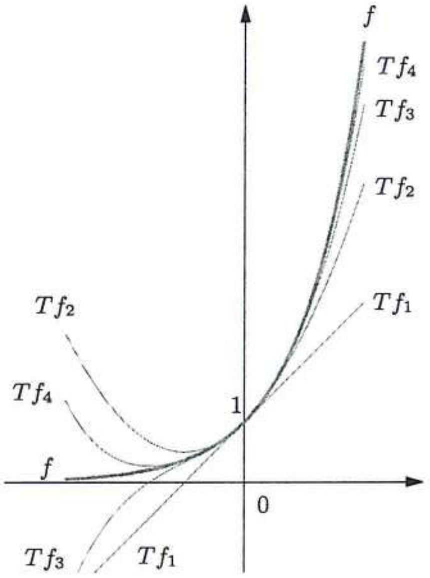

Figure 7.4. Local approximation of $f(x) = e^x$ by $Tf_n = Tf_{n,0}$ for $n = 1, 2, 3, 4$

Remark 7.4 Set $x = 1$ in the previous formula:

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^{\bar{x}}}{(n+1)!} \qquad (\text{con } 0 < \bar{x} < 1).$$

For any $n \geq 0$, we obtain an estimate (from below) of the number e, namely

$$e_n = \sum_{k=0}^{n} \frac{1}{k!}; \tag{7.11}$$

because $1 < e^{\bar{x}} < e < 3$ moreover, the following is an estimate of the error:

$$\frac{1}{(n+1)!} < e - e_n < \frac{3}{(n+1)!}.$$

In contrast to the sequence $\{a_n = \left(1 + \frac{1}{n}\right)^n\}$ used to define the constant e, the sequence $\{e_n\}$ converges at the rate of a factorial, hence very rapidly (compare Tables 7.1 and 3.1). Formula (7.11) gives therefore an excellent numerical approximation of the number e. □

The expansion of $f(x) = e^x$ at a generic x_0 follows from the fact that $f^{(k)}(x_0) = e^{x_0}$

$$e^x = e^{x_0} + e^{x_0}(x - x_0) + e^{x_0}\frac{(x - x_0)^2}{2} + \ldots + e^{x_0}\frac{(x - x_0)^n}{n!} + o\big((x - x_0)^n\big)$$
$$= \sum_{k=0}^{n} e^{x_0}\frac{(x - x_0)^k}{k!} + o\big((x - x_0)^n\big).$$

n	e_n
0	1.0000000000000
1	2.0000000000000
2	2.5000000000000
3	2.6666666666667
4	2.7083333333333
5	2.7166666666667
6	2.7180555555556
7	2.7182539682540
8	2.7182787698413
9	2.7182815255732
10	2.7182818011464

Table 7.1. Values of the sequence $\{e_n\}$ of (7.11)

The logarithm

The derivatives of the function $f(x) = \log x$ are

$$f'(x) = \frac{1}{x} = x^{-1}, \qquad f''(x) = (-1)x^{-2}, \qquad f'''(x) = (-1)(-2)x^{-3},$$

and in general,

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\, x^{-k}.$$

Thus for $k \geq 1$,

$$\frac{f^{(k)}(1)}{k!} = (-1)^{k-1}\frac{1}{k}$$

and the Taylor expansion of order n at $x_0 = 1$ is

$$\begin{aligned} \log x &= (x-1) - \frac{(x-1)^2}{2} + \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + o((x-1)^n) \\ &= \sum_{k=1}^{n}(-1)^{k-1}\frac{(x-1)^k}{k} + o((x-1)^n). \end{aligned} \tag{7.12}$$

Let us change the independent variable $x - 1 \to x$, to obtain the Maclaurin expansion of order n of $\log(1+x)$

$$\begin{aligned} \log(1+x) &= x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n) \\ &= \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} + o(x^n). \end{aligned} \tag{7.13}$$

The Maclaurin polynomials of order $n = 1, 2, 3, 4$ for $y = \log(1+x)$ are represented in Fig. 7.5.

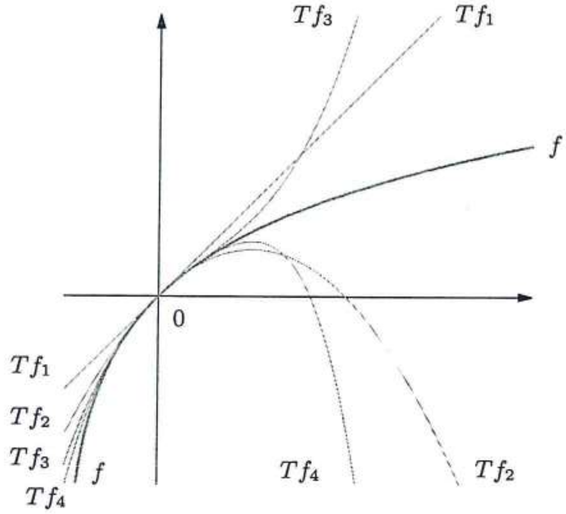

Figure 7.5. Local approximation of $f(x) = \log(1 + x)$ by $Tf_n = Tf_{n,0}$ for $n = 1, 2, 3, 4$

The trigonometric functions

The function $f(x) = \sin x$ is odd, so by Property 7.3 its Maclaurin expansion contains just odd powers of x. We have $f'(x) = \cos x$, $f'''(x) = -\cos x$ and in general $f^{(2k+1)}(x) = (-1)^k \cos x$, whence $f^{(2k+1)}(0) = (-1)^k$. Maclaurin's expansion up to order $n = 2m + 2$ reads

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2}) \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+2}). \end{aligned} \tag{7.14}$$

The typical structure of the expansion of an odd map should be noticed. Maclaurin's polynomial $Tf_{2m+2,0}$ of even order $2m + 2$ coincides with the polynomial $Tf_{2m+1,0}$ of odd degree $2m + 1$, for $f^{(2m+2)}(0) = 0$. Stopping at order $2m + 1$ would have rendered

$$\sin x = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+1}),$$

to which (7.14) is preferable, because it contains more information on the remainder's behaviour when $x \to 0$. Figure 7.6 represents the Maclaurin polynomials of degree $2m + 1$, $0 \le m \le 6$, of the sine.

As far as the even map $f(x) = \cos x$ is concerned, only even exponents appear. From $f''(x) = -\cos x$, $f^{(4)}(x) = \cos x$ and $f^{(2k)}(x) = (-1)^k \cos x$, it follows $f^{(2k)}(0) = (-1)^k$, so Maclaurin's expansion of order $n = 2m + 1$ is

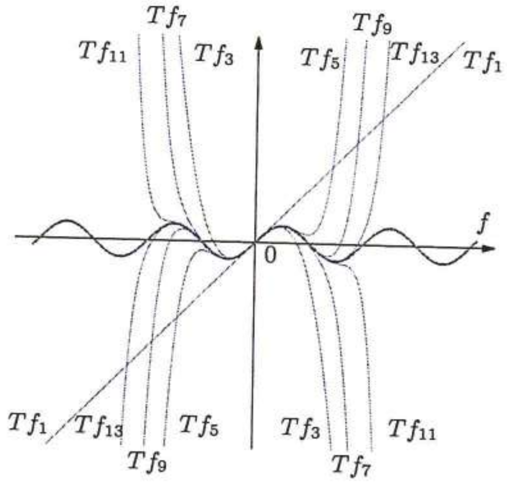

Figure 7.6. Local approximation of $f(x) = \sin x$ by polynomials $Tf_{2m+1} = Tf_{2m+1,0}$ with $0 \leq m \leq 6$

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1}) \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2m+1}).\end{aligned} \tag{7.15}$$

The considerations made about the sine apply also here. Maclaurin's polynomials of order $2m$ $(1 \leq m \leq 6)$ for $y = \cos x$ can be seen in Fig. 7.7.

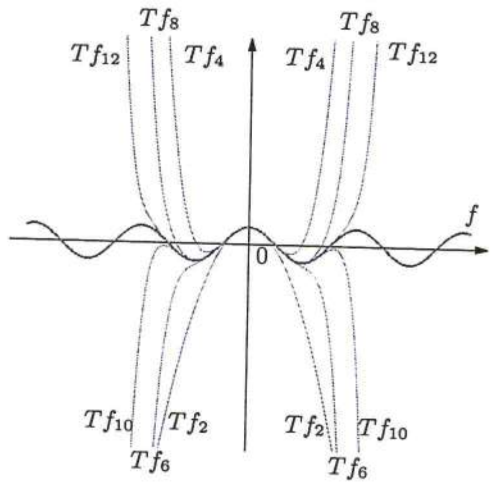

Figure 7.7. Local approximation of $f(x) = \cos x$ by $Tf_{2m} = Tf_{2m,0}$ when $1 \leq m \leq 6$

Power functions

Consider the family of maps $f(x) = (1+x)^\alpha$ for arbitrary $\alpha \in \mathbb{R}$. We have

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1} \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\ f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}. \end{aligned}$$

From the general relation $f^{(k)}(x) = \alpha(\alpha-1)\dots(\alpha-k+1)(1+x)^{\alpha-k}$ we get

$$f(0) = 1\,, \qquad \frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{for } k \ge 1\,.$$

At this point it becomes convenient to extend the notion of binomial coefficient (1.10), and allow α to be any real number by putting, in analogy to (1.11),

$$\binom{\alpha}{0} = 1\,, \qquad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{for } k \ge 1\,. \tag{7.16}$$

Maclaurin's expansion to order n is thus

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n) \\ &= \sum_{k=0}^{n}\binom{\alpha}{k}x^k + o(x^n). \end{aligned} \tag{7.17}$$

Let us see in detail what happens for special values of the parameter. When $\alpha = -1$

$$\begin{aligned} &\binom{-1}{2} = \frac{(-1)(-2)}{2} = 1, \quad \binom{-1}{3} = \frac{(-1)(-2)(-3)}{3!} = -1, \ \ldots, \\ &\binom{-1}{k} = \frac{(-1)(-2)\cdots(-k)}{k!} = (-1)^k, \end{aligned}$$

so

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n) = \sum_{k=0}^{n}(-1)^k x^k + o(x^n). \tag{7.18}$$

Choosing $\alpha = \frac{1}{2}$ gives

$$\binom{\frac12}{2} = \frac{\frac12(\frac12-1)}{2} = -\frac18, \qquad \binom{\frac12}{3} = \frac{\frac12(\frac12-1)(\frac12-2)}{3!} = \frac{1}{16},$$

and the expansion of $f(x) = \sqrt{1+x}$ arrested to the third order is

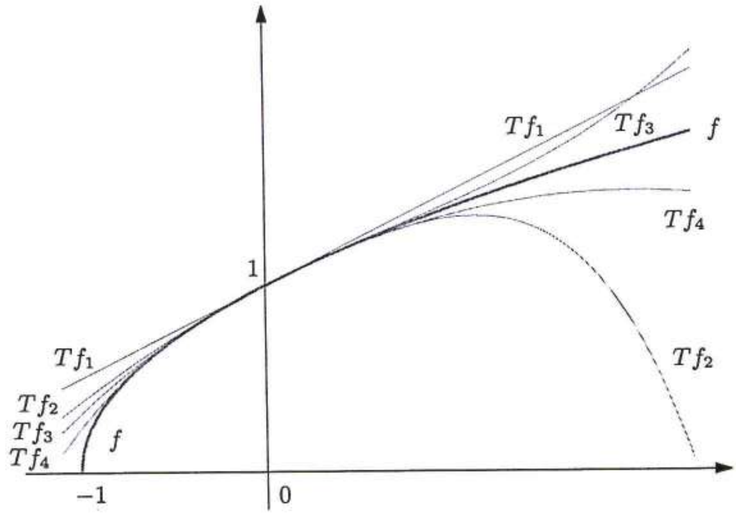

Figure 7.8. Local approximation of $f(x) = \sqrt{1+x}$ by $Tf_n = Tf_{n,0}$ for $n = 1, 2, 3, 4$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3).$$

The polynomials of order $n = 1, 2, 3, 4$ are shown in Fig. 7.8.

For conveniency, the following table collects the expansions with Peano's remainder obtained so far. A more comprehensive list is found on p. 476.

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

7.3 Operations on Taylor expansions

Consider the situation where a map f has a complicated analytic expression, that involves several elementary functions; it might not be that simple to find its Taylor expansion using the definition, because computing derivatives at a point up to a certain order n is no straighforward task. But with the expansions of the elementary functions at our avail, a more convenient strategy may be to start from these and combine them suitably to arrive at f. The techniques are explained in this section.

This approach is indeed justified by the following result.

Proposition 7.5 *Let $f : (a,b) \to \mathbb{R}$ be n times differentiable at $x_0 \in (a,b)$. If there exists a polynomial P_n, of degree $\leq n$, such that*

$$f(x) = P_n(x) + o\big((x-x_0)^n\big) \qquad \text{for } x \to x_0, \tag{7.19}$$

then P_n is the Taylor polynomial $T_n = Tf_{n,x_0}$ of order n for the map f at x_0.

Proof. Formula (7.19) is equivalent to

$$P_n(x) = f(x) + \varphi(x), \qquad \text{with } \varphi(x) = o\big((x-x_0)^n\big) \text{ for } x \to x_0.$$

On the other hand, Taylor's formula for f at x_0 reads

$$T_n(x) = f(x) + \psi(x), \qquad \text{with } \psi(x) = o\big((x-x_0)^n\big).$$

Therefore

$$P_n(x) - T_n(x) = \varphi(x) - \psi(x) = o\big((x-x_0)^n\big). \tag{7.20}$$

But the difference $P_n(x) - T_n(x)$ is a polynomial of degree lesser or equal than n, hence it may be written as

$$P_n(x) - T_n(x) = \sum_{k=0}^{n} c_k (x-x_0)^k.$$

The claim is that all coefficients c_k vanish. Suppose, by contradiction, there are some non-zero c_k, and let m be the smallest index between 0 and n such that $c_m \neq 0$. Then

$$P_n(x) - T_n(x) = \sum_{k=m}^{n} c_k (x-x_0)^k$$

so

$$\frac{P_n(x) - T_n(x)}{(x-x_0)^m} = c_m + \sum_{k=m+1}^{n} c_k (x-x_0)^{k-m},$$

by factoring out $(x-x_0)^m$. Taking the limit for $x \to x_0$ and recalling (7.20), we obtain

$$0 = c_m,$$

in contrast with the assumption. □

The proposition guarantees that however we arrive at an expression like (7.19) (in a mathematically correct way), this must be exactly the Taylor expansion of order n for f at x_0.

Example 7.6

Suppose the function $f(x)$ satisfies

$$f(x) = 2 - 3(x-2) + (x-2)^2 - \frac{1}{4}(x-2)^3 + o((x-2)^3) \qquad \text{for } x \to 2.$$

Then (7.7) implies

$$f(2) = 2, \quad f'(2) = -3, \quad \frac{f''(2)}{2} = 1, \quad \frac{f'''(2)}{3!} = -\frac{1}{4},$$

hence

$$f(2) = 2, \quad f'(2) = -3, \quad f''(2) = 2, \quad f'''(2) = -\frac{3}{2}.$$ □

For simplicity we shall assume henceforth $x_0 = 0$. This is always possible by a change of the variables, $x \to t = x - x_0$.

Let now

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n + o(x^n) = p_n(x) + o(x^n)$$

and

$$g(x) = b_0 + b_1 x + \ldots + b_n x^n + o(x^n) = q_n(x) + o(x^n)$$

be the Maclaurin expansions of the maps f and g.

Sums

From (5.5) a), it follows

$$\begin{aligned} f(x) \pm g(x) &= [p_n(x) + o(x^n)] \pm [q_n(x) + o(x^n)] \\ &= [p_n(x) \pm q_n(x)] + [o(x^n) \pm o(x^n)] \\ &= p_n(x) \pm q_n(x) + o(x^n). \end{aligned}$$

The expansion of a sum is the sum of the expansions involved.

Example 7.7

Let us find the expansions at the origin of the hyperbolic sine and cosine, introduced in Sect. 6.10.1. Changing x to $-x$ in

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{2n+2}}{(2n+2)!} + o(x^{2n+2})$$

gives

$$e^{-x} = 1 - x + \frac{x^2}{2} - \dots + \frac{x^{2n+2}}{(2n+2)!} + o(x^{2n+2}).$$

Thus

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}).$$

Similarly,

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}).$$

The analogies of these expansions to $\sin x$ and $\cos x$ should not go amiss. □

Note that when the expansions of f and g have the same monomial terms up to the exponent n, these *all cancel out* in the difference $f-g$. In order to find the first non-zero coefficient in the expansion of $f-g$ one has to look at an expansions of f and g of order $n' > n$. In general it is not possible to predict what the minimum n' will be, so one must proceed case by case. Using expansions 'longer' than necessary entails superfluous computations, but is no mistake, in principle. On the contrary, terminating an expansion 'too soon' leads to meaningless results or, in the worst scenario, to a wrong conclusion.

Example 7.8

Determine the order at 0 of

$$h(x) = e^x - \sqrt{1+2x}$$

by means of Maclaurin's expansion (see Sect. 7.4 in this respect).
Using first order expansions,

$$f(x) = e^x = 1 + x + o(x),$$
$$g(x) = \sqrt{1+2x} = 1 + x + o(x),$$

leads to the cancellation phenomenon just described. We may only say

$$h(x) = o(x),$$

which is clearly not enough for the order of h. Instead, if we expand to second order

$$f(x) = e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$
$$g(x) = \sqrt{1+2x} = 1 + x - \frac{x^2}{2} + o(x^2),$$

then

$$h(x) = x^2 + o(x^2)$$

shows $h(x)$ is infinitesimal of order two at the origin. □

Products

Using (5.5) d) and then (5.5) a) shows that

$$\begin{aligned} f(x)g(x) &= [p_n(x) + o(x^n)][q_n(x) + o(x^n)] \\ &= p_n(x)q_n(x) + p_n(x)o(x^n) + q_n(x)o(x^n) + o(x^n)o(x^n) \\ &= p_n(x)q_n(x) + o(x^n) + o(x^n) + o(x^{2n}) \\ &= p_n(x)q_n(x) + o(x^n). \end{aligned}$$

The product $p_n(x)q_n(x)$ contains powers of x larger than n; each of them is an $o(x^n)$, so we can eschew calculating it explicitly. We shall write

$$p_n(x)q_n(x) = r_n(x) + o(x^n),$$

intending that $r_n(x)$ gathers all powers of order $\leq n$, and nothing else, so in conclusion

$$f(x)g(x) = r_n(x) + o(x^n).$$

Example 7.9

Expand to second order

$$h(x) = \sqrt{1+x}\,\mathrm{e}^x$$

at the origin. Since

$$f(x) = \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2),$$
$$g(x) = \mathrm{e}^x = 1 + x + \frac{x^2}{2} + o(x^2),$$

it follows

$$\begin{aligned} h(x) &= \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)\left(1 + x + \frac{x^2}{2}\right) + o(x^2) \\ &= \left(1 + x + \frac{x^2}{2}\right) + \left(\frac{x}{2} + \frac{x^2}{2} + \boxed{\frac{x^3}{4}}\right) - \left(\frac{x^2}{8} + \boxed{\frac{x^3}{8}} + \boxed{\frac{x^4}{16}}\right) + o(x^2) \\ &= 1 + \frac{3}{2}x + \frac{7}{8}x^2 + o(x^2). \end{aligned}$$

The boxed terms have order larger than two, and therefore are already accounted for by the symbol $o(x^2)$. Because of this, they need not have been computed explicitly, although no harm was done. □

Quotients

Suppose $g(0) \neq 0$ and let

$$h(x) = \frac{f(x)}{g(x)},$$

for which we search an expansion

$$h(x) = r_n(x) + o(x^n), \qquad \text{with } r_n(x) = \sum_{k=0}^{n} c_k x^k.$$

From $h(x)g(x) = f(x)$ we have

$$r_n(x)q_n(x) + o(x^n) = p_n(x) + o(x^n).$$

This means that the part of degree $\leq n$ in the polynomial $r_n(x)q_n(x)$ (degree $2n$) must coincide with $p_n(x)$. By this observation we can determine the coefficients c_k of $r_n(x)$ starting from c_0. The practical computation may be carried out like the division algorithm for polynomials, so long as the latter are *ordered with respect to the increasing powers of* x:

$$
\begin{array}{r|l}
a_0 + a_1x + a_2x^2 + \dots + a_nx^n + o(x^n) & \underline{b_0 + b_1x + b_2x^2 + \dots + b_nx^n + o(x^n)} \\
\underline{a_0 + a'_1x + a'_2x^2 + \dots + a'_nx^n + o(x^n)} & c_0 + c_1x + \dots + c_nx^n + o(x^n) \\
0 + \tilde{a}_1x + \tilde{a}_2x^2 + \dots + \tilde{a}_nx^n + o(x^n) & \\
\underline{\tilde{a}_1x + \tilde{a}'_2x^2 + \dots + \tilde{a}'_nx^n + o(x^n)} & \\
\vdots & \\
\overline{0 + o(x^n)} &
\end{array}
$$

Examples 7.10

i) Let us compute the second order expansion of $h(x) = \dfrac{e^x}{3 + 2\log(1+x)}$. By (7.9), (7.13), we have $e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$, and $3 + 2\log(1+x) = 3 + 2x - x^2 + o(x^2)$; dividing

$$
\begin{array}{r|l}
1 + x + \frac{1}{2}x^2 + o(x^2) & \underline{3 + 2x - x^2 + o(x^2)} \\
\underline{1 + \frac{2}{3}x - \frac{1}{3}x^2 + o(x^2)} & \frac{1}{3} + \frac{1}{9}x + \frac{11}{54}x^2 + o(x^2) \\
\frac{1}{3}x + \frac{5}{6}x^2 + o(x^2) & \\
\underline{\frac{1}{3}x + \frac{2}{9}x^2 + o(x^2)} & \\
\frac{11}{18}x^2 + o(x^2) & \\
\underline{\frac{11}{18}x^2 + o(x^2)} & \\
o(x^2) &
\end{array}
$$

produces $h(x) = \frac{1}{3} + \frac{1}{9}x + \frac{11}{54}x^2 + o(x^2)$.

ii) Expand $h(x) = \tan x$ to the fourth order. The function being odd, it suffices to find Maclaurin's polynomial of degree three, which is the same as the one of order four. Since

$$\sin x = x - \frac{x^3}{6} + o(x^3) \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2} + o(x^3)\,,$$

dividing

Examples 7.11

i) Calculate to order two the expansion at 0 of

$$h(x) = \mathrm{e}^{\sqrt{1+x}-1}.$$

Define

$$f(x) = \sqrt{1+x} - 1 = \frac{x}{2} - \frac{x^2}{8} + o(x^2),$$

$$g(y) = \mathrm{e}^y = 1 + y + \frac{y^2}{2} + o(y^2).$$

Then

$$\begin{aligned} h(x) &= 1 + \left(\frac{x}{2} - \frac{x^2}{8} + o(x^2)\right) + \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{8} + o(x^2)\right)^2 + o(x^2) \\ &= 1 + \left(\frac{x}{2} - \frac{x^2}{8} + o(x^2)\right) + \frac{1}{2}\left(\frac{x^2}{4} + o(x^2)\right) + o(x^2) \\ &= 1 + \frac{x}{2} + o(x^2). \end{aligned}$$

ii) Expand to order three in 0 the map

$$h(x) = \frac{1}{1 + \log(1+x)}.$$

We can view this map as a quotient, but also as the composition of

$$f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

with

$$g(y) = \frac{1}{1+y} = 1 - y + y^2 - y^3 + o(y^3).$$

It follows

$$\begin{aligned} h(x) &= 1 - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) + \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right)^2 \\ &\quad - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right)^3 + o(x^3) \\ &= 1 - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) + \left(x^2 - x^3 + o(x^3)\right) - \left(x^3 + o(x^3)\right) + o(x^3) \\ &= 1 - x + \frac{3x^2}{2} - \frac{7x^3}{3} + o(x^3). \end{aligned}$$

□

Remark 7.12 If $f(x)$ is infinitesimal of order greater that one at the origin, we can spare ourselves some computations, in the sense that we might be able to infer the expansion of $h(x) = g\big(f(x)\big)$ of degree n from lower-order expansions of $g(y)$. For example, let f be infinitesimal of order 2 at the origin ($a_1 = 0,\ a_2 \neq 0$). Because $[f(x)]^k = a_2^k x^{2k} + o(x^{2k})$, an expansion for $g(y)$ of order $\frac{n}{2}$ (if n even) or $\frac{n+1}{2}$ (n odd) is sufficient to determine $h(x)$ up to degree n. (Note that $f(x)$ should be expanded to order n, in general.) □

Example 7.13

Expand to second order

$$h(x) = \sqrt{\cos x} = \sqrt{1 + (\cos x - 1)}.$$

Set

$$f(x) = \cos x - 1 = -\frac{x^2}{2} + o(x^2) \qquad \text{(2nd order)}$$

$$g(y) = \sqrt{1+y} = 1 + \frac{y}{2} + o(y) \qquad \text{(1st order)}.$$

Then

$$h(x) = 1 + \frac{1}{2}\left(-\frac{x^2}{2} + o(x^2)\right) + o(x^2)$$

$$= 1 - \frac{x^2}{4} + o(x^2) \qquad \text{(2nd order)}.$$

□

Asymptotic expansions (not of Taylor type)

In many situations where $f(x)$ is infinite for $x \to 0$ (or $x \to x_0$) it is possible to find an 'asymptotic' expansion of $f(x)$ in increasing powers of x $(x - x_0)$, by allowing negative powers in the expression:

$$f(x) = \frac{a_{-m}}{x^m} + \frac{a_{-m+1}}{x^{m-1}} + \ldots + \frac{a_{-1}}{x} + a_0 + a_1 x + \ldots + a_n x^n + o(x^n).$$

This form helps to understand better how f tends to infinity. In fact, if $a_{-m} \neq 0$, f will be infinite of order m with respect to the test function x^{-1}.

To a similar expansion one often arrives by means of the Taylor expansion of $\frac{1}{f(x)}$, which is infinitesimal for $x \to 0$.

We explain the procedure with an example.

Example 7.14

Let us expand 'asymptotically', for $x \to 0$, the function

$$f(x) = \frac{1}{e^x - 1}.$$

The exponential expansion arrested at order three gives

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

$$= x\left(1 + \frac{x}{2} + \frac{x^2}{6} + o(x^2)\right),$$

so

$$f(x) = \frac{1}{x}\frac{1}{1 + \frac{x}{2} + \frac{x^2}{6} + o(x^2)}.$$

The latter ratio can be treated using Maclaurin's formula

$$\frac{1}{1+y} = 1 - y + y^2 + o(y^2);$$

by putting

$$y = \frac{x}{2} + \frac{x^2}{6} + o(x^2)$$

in fact, we obtain

$$f(x) = \frac{1}{x}\left(1 - \frac{x}{2} + \frac{x^2}{12} + o(x^2)\right) = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} + o(x)\,,$$

the asymptotic expansion of f at the origin. Looking at such expression, we can deduce for instance that f is infinite of order 1 with respect to $\varphi(x) = \frac{1}{x}$, as $x \to 0$.

Ignoring the term $x/12$ and writing $f(x) = \frac{1}{x} - \frac{1}{2} + o(1)$ shows f is asymptotic to the hyperbola

$$g(x) = \frac{2-x}{2x}.$$

□

7.4 Local behaviour of a map via its Taylor expansion

Taylor expansions at a given point are practical tools for studying how a function locally behaves around that point. We examine in the sequel a few interesting applications of Taylor expansions.

Order and principal part of infinitesimal functions

Let

$$f(x) = a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n + o\big((x - x_0)^n\big)$$

be the Taylor expansion of order n at a point x_0, and suppose there is an index m with $1 \le m \le n$ such that

$$a_0 = a_1 = \ldots = a_{m-1} = 0, \qquad \text{but} \quad a_m \neq 0.$$

In a sufficiently small neighbourhood of x_0,

$$f(x) = a_m(x - x_0)^m + o\big((x - x_0)^m\big)$$

will behave like the polynomial

$$p(x) = a_m(x - x_0)^m,$$

which is the *principal part* of f with respect to the infinitesimal $y = x - x_0$. In particular, $f(x)$ has order m with respect to that test function.

Example 7.15

Compute the order of the infinitesimal $f(x) = \sin x - x\cos x - \frac{1}{3}x^3$ with respect to $\varphi(x) = x$ as $x \to 0$. Expanding sine and cosine with Maclaurin we have

$$f(x) = -\frac{1}{30}x^5 + o(x^5), \quad x \to 0.$$

Therefore f is infinitesimal of order 5 and has principal part $p(x) = -\frac{1}{30}x^5$. The same result descends from de l'Hôpital's Theorem, albeit differentiating five times is certainly more work than using well-known expansions. □

Local behaviour of a function

The knowledge of the Taylor expansion of f to order two around a point x_0,

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + o\big((x - x_0)^2\big)\,, \qquad x \to x_0\,,$$

allows us to deduce from (7.7) that

$$f(x_0) = a_0\,, \qquad f'(x_0) = a_1\,, \qquad f''(x_0) = 2a_2\,.$$

Suppose f is differentiable twice with continuity around x_0. By Theorem 4.2 the signs of a_0, a_1, a_2 (when $\neq 0$) coincide with the signs of $f(x)$, $f'(x)$, $f''(x)$, respectively, in a neighbourhood of x_0. This fact permits, in particular, to detect local monotonicity and convexity, because of Theorem 6.27 *b2)* and Corollary 6.38 *b2)*.

Example 7.6 (continuation)

Return to Example 7.6: we have $f(2) > 0$, $f'(2) < 0$ and $f''(2) > 0$. Around $x_0 = 2$ then, f is strictly positive, strictly decreasing and strictly convex. □

We deal with the cases $a_1 = 0$ or $a_2 = 0$ below.

Nature of critical points

Let x_0 be a critical point for f, which is assumed differentiable around x_0. By Corollary 6.28, different signs of f' at the left and right of x_0 mean that the point is an extremum; if the sign stays the same instead, x_0 is an inflection point with horizontal tangent.

When f possesses higher derivatives at x_0, in alternative to the sign of f' around x_0 we can understand what sort of critical point x_0 is by looking at the first non-zero derivative of f evaluated at the point. In fact,

Theorem 7.16 *Let f be differentiable $n \geq 2$ times at x_0 and suppose*

$$f'(x_0) = \ldots = f^{(m-1)}(x_0) = 0, \qquad f^{(m)}(x_0) \neq 0 \tag{7.21}$$

for some $2 \leq m \leq n$.

i) When m is even, x_0 is an extremum, namely a maximum if $f^{(m)}(x_0) < 0$, a minimum if $f^{(m)}(x_0) > 0$.

ii) When m is odd, x_0 is an inflection point with horizontal tangent; more precisely the inflection is descending if $f^{(m)}(x_0) < 0$, ascending if $f^{(m)}(x_0) > 0$.

Proof. Compare $f(x)$ and $f(x_0)$ around x_0. From (7.6)-(7.7) and the assumption (7.21), we have

$$f(x) - f(x_0) = \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m + o((x - x_0)^m).$$

But $o((x - x_0)^m) = (x - x_0)^m o(1)$, so

$$f(x) - f(x_0) = (x - x_0)^m \left[\frac{f^{(m)}(x_0)}{m!} + h(x)\right],$$

for a suitable $h(x)$, infinitesimal when $x \to x_0$. Therefore, in a sufficiently small neighbourhood of x_0, the term in square brackets has the same sign as $f^{(m)}(x_0)$, hence the sign of $f(x) - f(x_0)$, in that same neighbourhood, is determined by $f^{(m)}(x_0)$ and $(x - x_0)^m$. Examining all sign possibilities proves the claim. □

Example 7.17

Assume that around $x_0 = 1$ we have

$$f(x) = 2 - 15(x-1)^4 + 20(x-1)^5 + o((x-1)^5). \tag{7.22}$$

From this we deduce

$$f'(1) = f''(1) = f'''(1) = 0, \quad \text{but} \quad f^{(4)}(1) = -360 < 0.$$

Then x_0 is a relative maximum (Fig. 7.9, left).

Suppose now that in a neighbourhood of $x_1 = -2$ we can write

$$f(x) = 3 + 10(x+2)^5 - 35(x+2)^7 + o((x+2)^7). \tag{7.23}$$

Then

$$f'(-2) = f''(-2) = f'''(-2) = f^{(4)}(-2) = 0, \quad \text{and} \quad f^{(5)}(-2) = 10 \cdot 5! > 0,$$

telling x_1 is an ascending inflection with horizontal tangent (Fig. 7.9, right). □

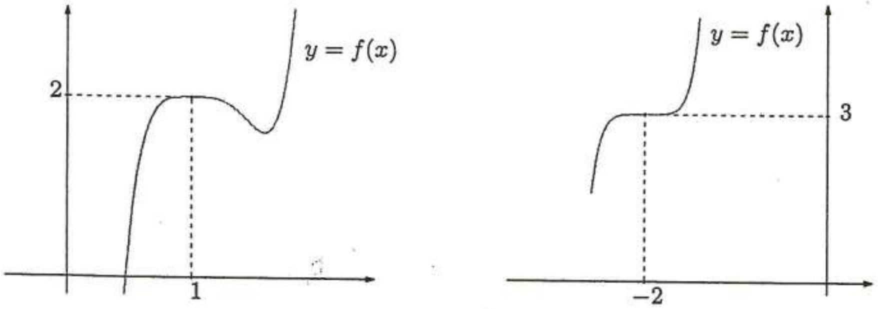

Figure 7.9. The map defined in (7.22), around $x_0 = 1$ (right), and the one defined in (7.23), around $x_0 = -2$ (left)

Points of inflection

Consider a twice differentiable f around x_0. By Taylor's formulas we can decide whether x_0 is an inflection point for f.

First though, we need to prove Corollary 6.39 stated in Chap. 6, whose proof we had to postpone to the present section.

Proof. a) Let x_0 be an inflection point for f. Denoting as usual by $y = t(x) = f(x_0) + f'(x_0)(x - x_0)$ the tangent line to f at x_0, Taylor's formula (7.6) $(n = 2)$ gives

$$f(x) - t(x) = \frac{1}{2}f''(x_0)(x - x_0)^2 + o\big((x - x_0)^2\big),$$

which we can write

$$f(x) - t(x) = (x - x_0)^2 \left[\frac{1}{2}f''(x_0) + h(x)\right]$$

for some infinitesimal h at x_0. By contradiction, if $f''(x_0) \neq 0$, in an arbitrarily small neighbourhood of x_0 the right-hand side would have constant sign at the left and right of x_0; this cannot be by hypothesis, as f is assumed to inflect at x_0.

b) In this case we use Taylor's formula (7.8) with $n = 2$. For any $x \neq x_0$, around x_0 there is a point $\bar{x}$, lying between x_0 and x, such that

$$f(x) - t(x) = \frac{1}{2}f''(\bar{x})(x - x_0)^2.$$

Analysing the sign of the right-hand side concludes the proof. □

Suppose, from now on, that $f''(x_0) = 0$ and f admits derivatives higher than the second. Instead of considering the sign of f'' around x_0, we may study the point x_0 by means of the first non-zero derivative of order > 2 evaluated at x_0.

Theorem 7.18 *Let f be n times differentiable $(n \geq 3)$ at x_0, with*

$$f''(x_0) = \ldots = f^{(m-1)}(x_0) = 0, \qquad f^{(m)}(x_0) \neq 0 \tag{7.24}$$

for some m $(3 \leq m \leq n)$.

i) When m is odd, x_0 is an inflection point: descending if $f^{(m)}(x_0) < 0$, ascending if $f^{(m)}(x_0) > 0$.

ii) When m is even, x_0 is not an inflection for f.

Proof. Just like in Theorem 7.16, we obtain

$$f(x) - t(x) = (x - x_0)^m \left[\frac{f^{(m)}(x_0)}{m!} + h(x)\right],$$

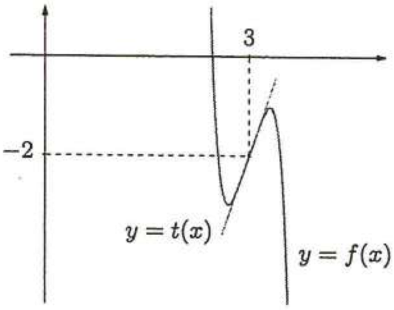

Figure 7.10. Local behaviour of the map (7.25)

where $h(x)$ is a suitable infinitesimal function for $x \to x_0$. The claim follows from a sign argument concerning the right-hand side. □

Example 7.19

Suppose that around $x_0 = 3$ we have

$$f(x) = -2 + 4(x-3) - 90(x-3)^5 + o\big((x-3)^5\big). \tag{7.25}$$

Then $f''(3) = f'''(3) = f^{(4)}(3) = 0$, $f^{(5)}(3) = -90 \cdot 5! < 0$. This implies that $x_0 = 3$ is a descending inflection for f (Fig. 7.10). □

7.5 Exercises

1. *Use the definition to write the Taylor polynomial, of order n and centred at x_0, for:*

   a) $f(x) = e^x, \quad n = 4, \quad x_0 = 2$
   b) $f(x) = \sin x, \quad n = 6, \quad x_0 = \frac{\pi}{2}$
   c) $f(x) = \log x, \quad n = 3, \quad x_0 = 3$
   d) $f(x) = \sqrt{2x+1}, \quad n = 3, \quad x_0 = 4$
   e) $f(x) = 7 + x - 3x^2 + 5x^3, \quad n = 2, \quad x_0 = 1$
   f) $f(x) = 2 - 8x^2 + 4x^3 + 9x^4, \quad n = 3, \quad x_0 = 0$

2. *Determine the Taylor expansion of the indicated functions of the highest-possible order; the expansion should be centred around x_0 and have Peano's remainder:*

   a) $f(x) = x^2|x| + e^{2x}, \quad x_0 = 0$
   b) $f(x) = 2 + x + (x-1)\sqrt[3]{x^2-1}, \quad x_0 = 1$

3. *With the aid of the elementary functions, write the Maclaurin expansion of the indicated functions of the given order, with Peano's remainder:*

   a) $f(x) = x\cos 3x - 3\sin x\,, \qquad n = 2$

   b) $f(x) = \log\dfrac{1+x}{1+3x}\,, \qquad n = 4$

   c) $f(x) = \mathrm{e}^{x^2}\sin 2x\,, \qquad n = 5$

   d) $f(x) = \mathrm{e}^{-x\cos x} + \sin x - \cos x\,, \qquad n = 2$

   e) $f(x) = \sqrt[3]{\cos(3x - x^2)}\,, \qquad n = 4$

   f) $f(x) = \dfrac{x}{\sqrt[6]{1+x^2}} - \sin x\,, \qquad n = 5$

   g) $f(x) = \cosh^2 x - \sqrt{1+2x^2}\,, \qquad n = 4$

   h) $f(x) = \dfrac{\mathrm{e}^{2x} - 1}{\sqrt{\cos 2x}}\,, \qquad n = 3$

   i) $f(x) = \dfrac{1}{-\sqrt{8}\sin x - 2\cos x}\,, \qquad n = 3$

   ℓ) $f(x) = \sqrt[3]{8 + \sin 24x^2} - 2(1 + x^2\cos x^2)\,, \qquad n = 4$

4. *Ascertain order and find principal part, for $x \to 0$, with respect to $\varphi(x) = x$ of the indicated functions:*

   a) $f(x) = \mathrm{e}^{\cos 2x} - \mathrm{e}$

   b) $f(x) = \dfrac{\cos 2x + \log(1 + 4x^2)}{\cosh 2x} - 1$

   c) $f(x) = \dfrac{\sqrt{x^3} - \sin^3\sqrt{x}}{\mathrm{e}^{3\sqrt{x}} - 1}$

   d) $f(x) = 2x + (x^2 - 1)\log\dfrac{1+x}{1-x}$

   e) $f(x) = x - \arctan\dfrac{x}{\sqrt{1-4x^2}}$

   f) $f(x) = \sqrt[3]{1-x^2} - \sqrt{1 - \dfrac{2}{3}x^2} + \sin\dfrac{x^4}{18}$

5. *Calculate order and principal part, when $x \to +\infty$, with respect to $\varphi(x) = \frac{1}{x}$ of the indicated functions:*

   a) $f(x) = \dfrac{1}{x-2} - \dfrac{1}{2(x-2) - \log(x-1)}$

   b) $f(x) = \mathrm{e}^{-\frac{x}{4x^2+1}} - 1$

   c) $f(x) = \sqrt[3]{1 + 3x^2 + x^3} - \sqrt[5]{2 + 5x^4 + x^5}$

   d) $f(x) = \sqrt[3]{2 + \sinh\dfrac{2}{x^2}} - \sqrt[3]{2}$

6. *Compute the limits:*

   a) $\displaystyle\lim_{x\to 0}(1 + x^6)^{1/(x^4\sin^2 3x)}$

   b) $\displaystyle\lim_{x\to 2}\frac{\cos\frac{3}{4}\pi x - \frac{3}{2}\pi\log\frac{x}{2}}{(4-x^2)^2}$

c) $\displaystyle\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{\sin(\tan x)}-\frac{1}{x}\right)$ d) $\displaystyle\lim_{x\to 0}\left(e^{x^7}+\sin^2 x-\sinh^2 x\right)^{1/x^4}$

e) $\displaystyle\lim_{x\to 0}\frac{18x^4}{\sqrt[3]{\cos 6x}-1+6x^2}$ f) $\displaystyle\lim_{x\to 0}\frac{3x^4[\log(1+\sinh^2 x)]\cosh^2 x}{1-\sqrt{1+x^3}\,\cos\sqrt{x^3}}$

7. *As a varies in $\mathbb{R}$, determine the order of the infinitesimal map*

$$h(x)=\log\cos x+\log\cosh(ax)$$

as $x\to 0$.

8. *Compute the sixth derivative of*

$$h(x)=\frac{\sinh(x^2+2\sin^4 x)}{1+x^{10}}$$

evaluated at $x=0$.

9. *Let*

$$\varphi(x)=\log(1+4x)-\sinh 4x+8x^2.$$

Determine the sign of $y=\sin\varphi(x)$ on a left and on a right neighbourhood of $x_0=0$.

10. *Prove that there exists a neighbourhood of 0 where*

$$2\cos(x+x^2)\le 2-x^2-2x^3.$$

11. *Compute the limit*

$$\lim_{x\to 0+}\frac{e^{x/2}-\cosh\sqrt{x}}{(x+\sqrt[5]{x})^\alpha}$$

for all values $\alpha\in\mathbb{R}^+$.

12. *Determine $\alpha\in\mathbb{R}$ so that*

$$f(x)=(\arctan 2x)^2-\alpha x\sin x$$

is infinitesimal of the fourth order as $x\to 0$.

7.5.1 Solutions

1. *Taylor's polynomials:*

a) All derivatives of $f(x)=e^x$ are identical with the function itself, so $f^{(k)}(2)=e^2$, $\forall k\ge 0$. Therefore

$$Tf_{4,2}(x)=e^2+e^2(x-2)+\frac{e^2}{2}(x-2)^2+\frac{e^2}{6}(x-2)^3+\frac{e^2}{24}(x-2)^4\,.$$

b) $Tf_{6,\frac{\pi}{2}}(x) = 1 - \frac{1}{2}(x-\frac{\pi}{2})^2 + \frac{1}{4!}(x-\frac{\pi}{2})^4 - \frac{1}{6!}(x-\frac{\pi}{2})^6.$

c) From $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ it follows $f(3) = \log 3$, $f'(3) = \frac{1}{3}$, $f''(3) = -\frac{1}{9}$, $f'''(3) = \frac{2}{27}$. Then

$$Tf_{3,3}(x) = \log 3 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3.$$

d) $Tf_{3,4}(x) = 3 + \frac{1}{3}(x-4) - \frac{1}{54}(x-4)^2 + \frac{1}{486}(x-4)^3.$

e) As $f'(x) = 1 - 6x + 15x^2$, $f''(x) = -6 + 30x$, we have $f(1) = 10$, $f'(1) = 10$, $f''(1) = 24$ and

$$Tf_{2,1}(x) = 10 + 10(x-1) + 12(x-1)^2.$$

Alternatively, we may substitute $t = x - 1$, i.e. $x = 1 + t$. The polynomial $f(x)$, written in the variable t, reads

$$g(t) = f(1+t) = 7 + (1+t) - 3(1+t)^2 + 5(1+t)^3 = 10 + 10t + 12t^2 + 5t^3.$$

Therefore the Taylor polynomial of $f(x)$ centred at $x_0 = 1$ corresponds to the Maclaurin polynomial of $g(t)$, whence immediately

$$Tg_{2,0}(t) = 10 + 10t + 12t^2.$$

Returning to the variable x, we find the same result.

f) $Tf_{3,0}(x) = 2 - 8x^2 + 4x^3.$

2. *Taylor's expansions:*

a) We can write $f(x) = g(x) + h(x)$ using $g(x) = x^2|x|$ and $h(x) = e^{2x}$. The sum $h(x)$ is differentiable on $\mathbb{R}$ *ad libitum*, whereas $g(x)$ is continuous on $\mathbb{R}$ but arbitrarily differentiable only for $x \neq 0$. Additionally

$$g'(x) = \begin{cases} 3x^2 & \text{if } x > 0\,, \\ -3x^2 & \text{if } x < 0\,, \end{cases} \qquad g''(x) = \begin{cases} 6x & \text{if } x > 0\,, \\ -6x & \text{if } x < 0\,, \end{cases}$$

so

$$\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^-} g'(x) = 0, \qquad \lim_{x\to 0^+} g''(x) = \lim_{x\to 0^-} g''(x) = 0.$$

By Theorem 6.15 we infer g is differentiable twice at the origin, with vanishing derivatives. Since $g''(x) = 6|x|$ is not differentiable at $x = 0$, g is not differentiable three times at 0, which makes f expandable only up to order 2. From $h'(x) = 2e^{2x}$ and $h''(x) = 4e^{2x}$, we have $f(0) = 1$, $f'(0) = 2$, $f''(0) = 4$, so Maclaurin's formula reads:

$$f(x) = 1 + 2x + 2x^2 + o(x^2)\,.$$

b) The map is differentiable only once at $x_0 = 1$, and the expansion is $f(x) = 3 + (x-1) + o(x-1)$.

3. *Maclaurin's expansions:*

a) $f(x) = -2x + o(x^2)$.

b) Writing $f(x) = \log(1+x) - \log(1+3x)$, we can use the expansion of $\log(1+t)$ with $t = x$ and $t = 3x$

$$\begin{aligned} f(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - 3x + \frac{(3x)^2}{2} - \frac{(3x)^3}{3} + \frac{(3x)^4}{4} + o(x^4) \\ &= -2x + 4x^2 - \frac{26}{3}x^3 + 20x^4 + o(x^4)\,. \end{aligned}$$

c) Combining the expansions of e^t with $t = x^2$, and of $\sin t$ with $t = 2x$:

$$\begin{aligned} f(x) &= \left(1 + x^2 + \frac{x^4}{2} + o(x^5)\right)\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + o(x^5)\right) \\ &= 2x + 2x^3 + x^5 - \frac{4}{3}x^3 - \frac{4}{3}x^5 + \frac{4}{15}x^5 + o(x^5) \\ &= 2x + \frac{2}{3}x^3 - \frac{1}{15}x^5 + o(x^5)\,. \end{aligned}$$

d) $f(x) = x^2 + o(x^2)$.

e) $f(x) = 1 - \frac{3}{2}x^2 + x^3 - \frac{31}{24}x^4 + o(x^4)$.

f) This is solved by expanding $(1+t)^\alpha$ and changing $\alpha = -\frac{1}{6}$ and $t = x^2$:

$$\begin{aligned} \frac{x}{\sqrt[6]{1+x^2}} &= x\left(1+x^2\right)^{-1/6} = x\left(1 - \frac{1}{6}x^2 + \binom{-\frac{1}{6}}{2}x^4 + o(x^4)\right) \\ &= x - \frac{1}{6}x^3 + \frac{7}{72}x^5 + o(x^5), \end{aligned}$$

from which

$$f(x) = x - \frac{1}{6}x^3 + \frac{7}{72}x^5 - x + \frac{1}{6}x^3 - \frac{1}{5!}x^5 + o(x^5) = \frac{4}{45}x^5 + o(x^5).$$

g) Referring to the expansions of $\cosh x$ and $(1+t)^\alpha$, with $\alpha = \frac{1}{2}$, $t = 2x^2$:

$$\begin{aligned} f(x) &= \left(1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + o(x^4)\right)^2 - \left(1+2x^2\right)^{1/2} \\ &= 1 + x^2 + \frac{1}{4}x^4 + \frac{2}{4!}x^4 + o(x^4) - \left(1 + \frac{1}{2}2x^2 + \binom{1/2}{2}(2x^2)^2 + o(x^4)\right) \\ &= 1 + x^2 + \frac{1}{3}x^4 - 1 - x^2 + \frac{1}{2}x^4 + o(x^4) = \frac{5}{6}x^4 + o(x^4)\,. \end{aligned}$$

h) $f(x) = 2x + 2x^2 + \frac{10}{3}x^3 + o(x^3)$.

i) Substitute to $\sin x$, $\cos x$ the respective Maclaurin expansions, to the effect that
$$f(x) = \frac{1}{-2-\sqrt{8}x+x^2+\frac{\sqrt{8}}{3!}x^3+o(x^3)}.$$
Expansion of the reciprocal eventually gives us
$$f(x) = -\frac{1}{2}+\frac{\sqrt{2}}{2}x-\frac{5}{4}x^2+\frac{17}{12}\sqrt{2}\,x^3+o(x^3).$$

ℓ) $f(x) = -2x^4 + o(x^4)$.

4. *Order of infinitesimal and principal part for $x \to 0$:*

a) The order is 2 and $p(x) = -2\,\mathrm{e}\,x^2$ the principal part.

b) Write
$$h(x) = \frac{\cos 2x + \log(1+4x^2) - \cosh 2x}{\cosh 2x},$$
and note that the order for $x \to 0$ can be deduced from the numerator only, for the denominator converges to 1. The expansions of $\cos t$, $\log(1+t)$ and $\cosh t$ are known, so
$$\begin{aligned}&\cos 2x + \log(1+4x^2) - \cosh 2x \\ &= 1-\frac{1}{2}(2x)^2+\frac{1}{4!}(2x)^4+(2x)^2-\frac{1}{2}(2x)^4-1-\frac{1}{2}(2x)^2-\frac{1}{4!}(2x)^4+o(x^4) \\ &= -8x^4+o(x^4).\end{aligned}$$
Thus the order is 4, the principal part $p(x) = -8x^4$.

c) Expanding $\sin t$ and e^t, then putting $t = \sqrt{x}$, we have
$$g(t) = \frac{t^3-\sin^3 t}{\mathrm{e}^{3t}-1} = \frac{t^3-\left(t-\frac{1}{6}t^3+o(t^3)\right)^3}{1+3t+o(t)-1} = \frac{\frac{1}{2}t^5+o(t^5)}{3t+o(t)} = \frac{1}{6}t^4+o(t^4)$$
for $t \to 0$. Hence
$$f(x) = \frac{1}{6}x^2+o(x^2),$$
implying that the order is 2 and $p(x) = \frac{1}{6}x^2$.

d) The map has order 3 with principal part $p(x) = \frac{4}{3}x^3$.

e) Use the expansion of $(1+t)^\alpha$ (where $\alpha = -\frac{1}{2}$) and $\arctan t$:
$$(1-4x^2)^{-1/2} = 1+2x^2+o(x^3), \qquad \frac{x}{\sqrt{1-4x^2}} = x+2x^3+o(x^4)$$
$$\begin{aligned}\arctan\frac{x}{\sqrt{1-4x^2}} &= x+2x^3+o(x^4)-\frac{1}{3}(x-2x^3+o(x^4))^3+o(x^3) \\ &= x+\frac{5}{3}x^3+o(x^3).\end{aligned}$$

In conclusion,

$$f(x) = -\frac{5}{3}x^3 + o(x^3),,$$

so that the order is 3 and the principal part $p(x) = -\frac{5}{3}x^3$.

f) Order 6 and principal part $p(x) = (-\frac{5}{3^4} + \frac{1}{2\cdot 3^3})x^6$.

5. *Order of infinitesimal and principal part as $x \to +\infty$:*

a) When $x \to +\infty$ we write

$$\begin{aligned} f(x) &= \frac{x-2-\log(x-1)}{2(x-2)^2-(x-2)\log(x-1)} \\ &= \frac{x-2-\log(x-1)}{2x^2-8x+8-(x-2)\log(x-1)} \\ &= \frac{x+o(x)}{2x^2+o(x^2)} = \frac{1}{2x} + o\left(\frac{1}{x}\right), \end{aligned}$$

from which one can recognise the order 1 and the principal part $p(x) = \frac{1}{2x}$.

b) The map is infinitesimal of order one, with principal part $p(x) = -\frac{1}{4x}$.

c) Write

$$\begin{aligned} f(x) &= \sqrt[3]{x^3\left(1+\frac{3}{x}+\frac{1}{x^3}\right)} - \sqrt[5]{x^5\left(1+\frac{5}{x}+\frac{2}{x^5}\right)} \\ &= x\left(1+\frac{3}{x}+\frac{1}{x^3}\right)^{1/3} - x\left(1+\frac{5}{x}+\frac{2}{x^5}\right)^{1/5}. \end{aligned}$$

Using the expansion of $(1+t)^\alpha$ first with $\alpha = \frac{1}{3}$, $t = \frac{3}{x} + \frac{1}{x^3}$, then with $\alpha = \frac{1}{5}$, $t = \frac{5}{x} + \frac{2}{x^5}$, we get

$$\begin{aligned} f(x) &= x\left[1 + \frac{1}{3}\left(\frac{3}{x}+\frac{1}{x^3}\right) - \binom{\frac{1}{3}}{2}\left(\frac{3}{x}+\frac{1}{x^3}\right)^2 + o\left(\frac{1}{x^2}\right) + \right. \\ &\qquad \left. -1 - \frac{1}{5}\left(\frac{5}{x}+\frac{2}{x^5}\right) - \binom{\frac{1}{5}}{2}\left(\frac{5}{x}+\frac{2}{x^5}\right)^2 + o\left(\frac{1}{x^2}\right)\right] \\ &= x\left(\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{x^2} - \frac{1}{x} - \frac{2}{5x^5} + \frac{2}{x^2} + o\left(\frac{1}{x^2}\right)\right) \\ &= x\left(\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right) = \frac{1}{x} + o\left(\frac{1}{x}\right). \end{aligned}$$

Therefore the order is 1, and $p(x) = \frac{1}{x}$.

d) The order is 2 and $p(x) = \frac{\sqrt[3]{2}}{3x^2}$.

6. *Limits:*

a) Let us rewrite as

$$\lim_{x\to 0}(1+x^6)^{1/(x^4\sin^2 3x)} = \lim_{x\to 0}\exp\left(\frac{1}{x^4\sin^2 3x}\ \log(1+x^6)\right)$$
$$= \exp\left(\lim_{x\to 0}\frac{\log(1+x^6)}{x^4\sin^2 3x}\right) = \mathrm{e}^L.$$

To compute L, take the expansions of $\log(1+t)$ and $\sin t$:

$$L = \lim_{x\to 0}\frac{x^6+o(x^6)}{x^4(3x+o(x^2))^2} = \lim_{x\to 0}\frac{x^6+o(x^6)}{9x^6+o(x^6)} = \frac{1}{9}.$$

The required limit is $\mathrm{e}^{1/9}$.

b) $\frac{3}{256}\pi$.

c) Expanding the sine and tangent,

$$L = \lim_{x\to 0}\frac{x-\sin(\tan x)}{x^2\sin(\tan x)} = \lim_{x\to 0}\frac{x-\tan x+\frac{1}{6}\tan^3 x+o(x^3)}{x^2(\tan x+o(x))}$$
$$= \lim_{x\to 0}\frac{x-x-\frac{1}{3}x^3+\frac{1}{6}x^3+o(x^3)}{x^3+o(x^3)} = \lim_{x\to 0}\frac{-\frac{1}{6}x^3+o(x^3)}{x^3+o(x^3)} = -\frac{1}{6}.$$

d) $\mathrm{e}^{-2/3}$; e) -1.

f) Observe that

$$3x^4[\log(1+\sinh^2 x)]\cosh^2 x \sim 3x^4\sinh^2 x \sim 3x^6\,.$$

for $x\to 0$. Moreover, the denominator can be written as

$$\text{Den}\ :\ 1-(1+x^3)^{1/2}\cos x^{3/2}$$
$$= 1-\left(1+\frac{1}{2}x^3+\binom{1/2}{2}x^6+o(x^6)\right)\left(1-\frac{1}{2}x^3+\frac{1}{4!}x^6+o(x^6)\right)$$
$$= 1-\left(1+\frac{1}{2}x^3-\frac{1}{8}x^6-\frac{1}{2}x^3-\frac{1}{4}x^6+\frac{1}{24}x^6+o(x^6)\right) = \frac{1}{3}x^6+o(x^6).$$

The limit is thus

$$\lim_{x\to 0}\frac{3x^6+o(x^6)}{\frac{1}{3}x^6+o(x^6)} = 9.$$

7. Expand $\log(1+t)$, $\cos t$, $\cosh t$, so that

$$h(x) = \log\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^5)\right)+\log\left(1+\frac{1}{2}(ax)^2+\frac{1}{4!}(ax)^4+o(x^5)\right)$$

6. *Limits:*

a) Let us rewrite as

$$\lim_{x\to 0}(1+x^6)^{1/(x^4\sin^2 3x)} = \lim_{x\to 0}\exp\left(\frac{1}{x^4\sin^2 3x}\;\log(1+x^6)\right)$$
$$= \exp\left(\lim_{x\to 0}\frac{\log(1+x^6)}{x^4\sin^2 3x}\right) = \mathrm{e}^L.$$

To compute L, take the expansions of $\log(1+t)$ and $\sin t$:

$$L = \lim_{x\to 0}\frac{x^6+o(x^6)}{x^4(3x+o(x^2))^2} = \lim_{x\to 0}\frac{x^6+o(x^6)}{9x^6+o(x^6)} = \frac{1}{9}.$$

The required limit is $\mathrm{e}^{1/9}$.

b) $\frac{3}{256}\pi$.

c) Expanding the sine and tangent,

$$L = \lim_{x\to 0}\frac{x-\sin(\tan x)}{x^2\sin(\tan x)} = \lim_{x\to 0}\frac{x-\tan x+\frac{1}{6}\tan^3 x+o(x^3)}{x^2(\tan x+o(x))}$$
$$= \lim_{x\to 0}\frac{x-x-\frac{1}{3}x^3+\frac{1}{6}x^3+o(x^3)}{x^3+o(x^3)} = \lim_{x\to 0}\frac{-\frac{1}{6}x^3+o(x^3)}{x^3+o(x^3)} = -\frac{1}{6}.$$

d) $\mathrm{e}^{-2/3}$; e) -1.

f) Observe that

$$3x^4[\log(1+\sinh^2 x)]\cosh^2 x \sim 3x^4\sinh^2 x \sim 3x^6\,.$$

for $x\to 0$. Moreover, the denominator can be written as

$$\text{Den}\;:\;1-(1+x^3)^{1/2}\cos x^{3/2}$$
$$= 1-\left(1+\frac{1}{2}x^3+\binom{1/2}{2}x^6+o(x^6)\right)\left(1-\frac{1}{2}x^3+\frac{1}{4!}x^6+o(x^6)\right)$$
$$= 1-\left(1+\frac{1}{2}x^3-\frac{1}{8}x^6-\frac{1}{2}x^3-\frac{1}{4}x^6+\frac{1}{24}x^6+o(x^6)\right) = \frac{1}{3}x^6+o(x^6).$$

The limit is thus

$$\lim_{x\to 0}\frac{3x^6+o(x^6)}{\frac{1}{3}x^6+o(x^6)} = 9.$$

7. Expand $\log(1+t)$, $\cos t$, $\cosh t$, so that

$$h(x) = \log\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^5)\right)+\log\left(1+\frac{1}{2}(ax)^2+\frac{1}{4!}(ax)^4+o(x^5)\right)$$

$$= -\frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{4!}x^4\right)^2 + o(x^5) + \frac{a^2}{2}x^2 + \frac{a^4}{4!}x^4 - \\ -\frac{1}{2}\left(\frac{a^2}{2}x^2 + \frac{a^4}{4!}x^4\right)^2 + o(x^5)$$

$$= \frac{1}{2}(a^2-1)x^2 + \left(\frac{1}{4!} - \frac{1}{8}\right)(a^4+1)x^4 + o(x^5).$$

If $a \neq \pm 1$, $h(x)$ is infinitesimal of order 2 for $x \to 0$. If $a = \pm 1$ the first non-zero coefficient multiplies x^4, making h infinitesimal of order 4 for $x \to 0$.

8. In order to compute $h^{(6)}(x)$ at $x = 0$ we use the fact that the Maclaurin coefficient of x^6 is $a_6 = \frac{h^{(6)}(0)}{6!}$. Therefore we need the expansion up to order six. Working on $\sin t$ and $\sinh t$, the numerator of h becomes

$$\text{Num} : \sinh\left(x^2 + 2\left(x^4 - \frac{4}{3!}x^6 + o(x^6)\right)\right)$$

$$= \sinh\left(x^2 + 2x^4 - \frac{4}{3}x^6 + o(x^6)\right) = x^2 + 2x^4 - \frac{4}{3}x^6 + \frac{1}{3!}x^6 + o(x^6)$$

$$= x^2 + 2x^4 - \frac{7}{6}x^6 + o(x^6).$$

Dividing $x^2 + 2x^4 - \frac{7}{6}x^6 + o(x^6)$ by $1 + x^{10}$ one finds

$$h(x) = x^2 + 2x^4 - \frac{7}{6}x^6 + o(x^6),$$

so $h^{(6)}(0) = -\frac{7}{6} \cdot 6! = -840$.

9. Use the expansions of $\log(1+t)$ and $\sinh t$ to write

$$\varphi(x) = 4x - \frac{1}{2}(4x)^2 + \frac{1}{3}(4x)^3 - 4x - \frac{1}{3!}(4x)^3 + 8x^2 + o(x^3) = \frac{32}{3}x^3 + o(x^3).$$

Since the sine has the same sign as its argument around the origin, the function $y = \sin\varphi(x)$ is negative for $x < 0$ and positive for $x > 0$.

10. Using $\cos t$ in Maclaurin's form,

$$2\cos(x+x^2) = 2\left(1 - \frac{1}{2}(x+x^2)^2 + \frac{1}{4!}(x+x^2)^4 + o\left((x+x^2)^4\right)\right)$$

$$= 2 - (x^2 + 2x^3 + x^4) + \frac{1}{3 \cdot 4}x^4 + o(x^4)$$

$$= 2 - x^2 - 2x^3 - \frac{11}{12}x^4 + o(x^4)$$

on some neighbourhood I of the origin. Then the given inequality holds on I, because the principal part of the difference between right- and left-hand side, clearly negative, equals $-\frac{11}{12}x^4$.

11. Expand numerator and denominator separately as

$$\begin{aligned}
\text{Num}\ :\ & 1+\frac{1}{2}x+\frac{1}{2}\left(\frac{x}{2}\right)^2+o(x^2)-\left(1+\frac{1}{2}x+\frac{1}{4!}x^2+o(x^2)\right)\\
&=\left(\frac{1}{8}-\frac{1}{4!}\right)x^2+o(x^2)=\frac{1}{12}x^2+o(x^2)\,,\\
\text{Den}\ :\ & \left[x^{1/5}\left(1+x^{4/5}\right)\right]^\alpha\\
&=x^{\alpha/5}\left(1+x^{4/5}\right)^\alpha=x^{\alpha/5}\left(1+\alpha x^{4/5}+o(x^{4/5})\right).
\end{aligned}$$

Then

$$\lim_{x\to 0+}\frac{e^{x/2}-\cosh\sqrt{x}}{(x+\sqrt[5]{x})^\alpha}=\lim_{x\to 0+}\frac{\frac{1}{12}x^2+o(x^2)}{x^{\alpha/5}\left(1+\alpha x^{4/5}+o(x^{4/5})\right)}$$

$$=\begin{cases}\frac{1}{12} & \text{if } 2=\frac{\alpha}{5}\,,\\ 0 & \text{if } 2>\frac{\alpha}{5}\,,\\ +\infty & \text{if } 2<\frac{\alpha}{5}\end{cases}=\begin{cases}\frac{1}{12} & \text{if } \alpha=10\,,\\ 0 & \text{if } \alpha<10\,,\\ +\infty & \text{if } \alpha>10.\end{cases}$$

12. Writing $\arctan t$ and $\sin t$ in Maclaurin's form provides

$$\begin{aligned}
f(x)&=\left(2x-\frac{1}{3}(2x)^3+o(x^3)\right)^2-\alpha x\left(x-\frac{1}{6}x^3+o(x^3)\right)\\
&=4x^2-\frac{32}{3}x^4+o(x^4)-\alpha x^2+\frac{\alpha}{6}x^4+o(x^4)\\
&=(4-\alpha)x^2-\left(\frac{32}{3}-\frac{\alpha}{6}\right)x^4+o(x^4).
\end{aligned}$$

This proves $f(x)$ infinitesimal of the fourth order at the origin if $\alpha=4$. For such value in fact,

$$f(x)=10x^4+o(x^4).$$