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POLYCOPIE

Signal processing
“Traitement du Signal”

Level: Master 1
TAIP -RSD-IAA
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Preamble

This signal processing course is intended for first-year computer science master's students specializing in TAIP RSD AND IAA. It correspond to the official program of the “signal processing” module whose fundamental objective is the “mathematical description” of signals. This convenient representation of the signal makes it possible to highlight its main characteristics (frequency distribution, energy, etc.) and to analyze the modifications undergone during the transmission or processing of these signals.

This manual, written with a constant concern for simplicity, is structured into Seven chapters. Are devoted to the basics of analog signal theory. The transformation of analog signals into digital signals At the end, some exercises are proposed.

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CHAPTER I

General information about signals

Introduction

The signal processing is a technical discipline which aims to develop, detect and interpret signals carrying information. This discipline is based on signal theory which gives a mathematical description of signals. This theory essentially uses linear algebra, functional analysis, electricity and the study of random processes.

Historically, signal processing appeared at the beginning of the 20th century, at the same time as electronics (FLEMING, 1905, detection and amplification of weak signals). However, we can note the first work in the 19th century with the invention of the electric telegraph (MORSE, COOKE, WHEATSTONE, 1830), the telephone (BELL, 1876) and the radio (MARCONI,POPOV, 1895).

Signal theory appeared in 1930 with the first work of WIENER and KINTCHINE on random processes, and that of NYQUIST and HARTLEY on the quantity of information transmitted on a telegraphic message. The essential contributions to signal processing and signal theory only came after the Second World War. Invention of the transistor in 1948, work by SHANNON on communication, WIENER on optimal filtering and SCHWARTZ on distributions.

Signal processing has become an essential science these days: all measurement and information processing applications implement processing techniques on the signal to extract the desired information. Initially intended to extract the signal from noise during measurements (sensors), signal processing is widely applied in telecommunications in diverse and varied applications. We can cite:

- Information protection against noise such as techniques to reduce the rate
error or to counter the effects of the channel (equalization technique).

- The development of electronic applications and easy evolution towards new ones features such as selective filtering, the implementation of various modulation/ demodulation techniques, etc.

The improvement in system performance in recent years is due, in large part, to the application of signal processing techniques. This is particularly the case in medical imaging, telephony and telecommunications. The hardware structures are essentially the same, but the signal processing techniques use sophisticated digital processing. The implications regarding a medical diagnosis, the surveillance of an aerial or underwater area or even the location of faults are immediate. The objective of signal processing then appears as a mathematical tool used to extract as much useful information as possible from a signal disturbed by noise. Useful signals are often disrupted by parasitic signals (noise) which sometimes mask them completely. To attenuate, if not eliminate, this noise, it is necessary to know its characteristics as well as those of the useful signal. This is why signal processing is a very mathematical discipline. The techniques used can be applied to an analog (continuous) signal but given their complexity, digital processing is almost always necessary. It is made possible thanks to the power of calculation circuits and modern computers.

1. From signal theory to signal processing

The words signal And information are common in everyday language. In the scientific world, these words have very specific meanings: in particular, information theory, signal theory and signal processing correspond to different notions, illustrated in Figure 1.1 in the context of a chain of communications. Even more generally:

Signal theory represents the set of mathematical tools describing the signals and noises emitted by a source or modified by a processing system, in particular: the Fourier transform (TF), Functional analysis, Statistical analysis and Estimation method and which makes it possible to describe signals and noises emitted by a source, or modified by a processing system (figure 1.1). The fundamental objective of signal theory is the mathematical description of signals. It thus makes it possible to establish a representation of a signal according to time or space containing information to be stored, transformed, transmitted or received. Signal theory does not prejudge the physical nature of the signal,

Information theory is the set of mathematical tools that makes it possible to describe the transmission of messages conveyed from a source to a recipient,

Signal processing is the set of methods and algorithms that make it possible to develop or interpret signals carrying information. More precisely :

- development: synthesis, analysis, regeneration, coding, modulation, identification,

frequency change,

- interpretation: decoding, demodulation, filtering, detection, identification, measurement, etc.

Currently, processing methods are almost entirely digital, which implies:

- temporal sampling, and representation of the signals in discrete time,

- digitization of the signal by analog/digital conversion, which involves quantification of the signal.

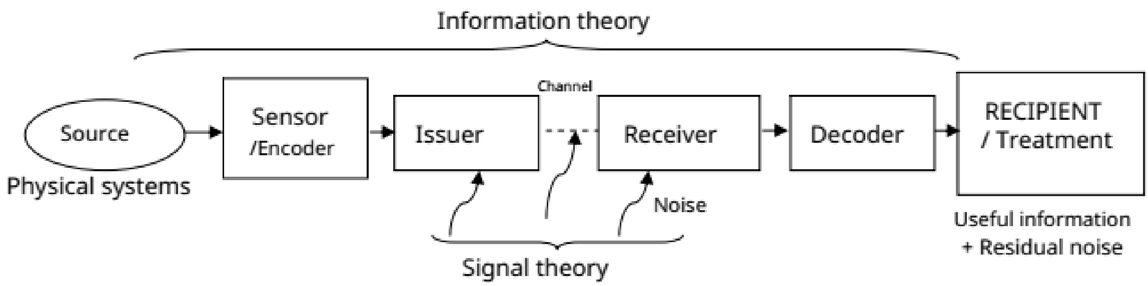


Figure 1.1. Information and signal theory in an analog signal transmission chain.

Signal theory provides the description and mathematical tools (or modeling) necessary to manipulate deterministic or random signals, that is, to describe, characterize and compare them. It provides the means to highlight, in convenient mathematical form, the main characteristics of a signal: the spectral distribution of its energy or the statistical distribution of its amplitude for example, the classification of signals and their description in vector spaces, called Hilbert spaces. Signal processing is the technical discipline which, based on signal and information theory, the resources of electronics, computer science and applied physics, aims to develop or interpretation of signals carrying information. It finds its application in all areas concerned with the perception, transmission or use of this information, as soon as a sensor measures a physical quantity carrying information, which is disturbed (by noise or the measurement system) and which must be processed to extract useful information. Signal processing methods make it possible to imagine safer, more reliable, faster methods for analyzing and transmitting signals. It also offers the means to analyze the nature of alterations or modifications undergone by signals as they pass through functional blocks (generally electrical or electronic devices). The mathematical description of signals makes it possible to design and characterize processing systems. some information. Noise will represent any “signal” or disturbing phenomenon.

2. Signal and noise

2.1. Definition of a Signal

comes from Latin signum : sign ; variation of a physical quantity of any nature carrying information.

A signal is therefore the physical representation of the information it carries from its source to its destination. Its physical nature can vary greatly: acoustic, electronic, optical, etc. It constitutes a physical manifestation of a measurable quantity (current, voltage, force, temperature, pressure, etc.). The signals considered in this course are time-dependent signals obtained using sensors. Signal theory remains valid regardless of the physical nature of the signal.

Word signal is almost always associated with the word noise. The latter is used in common language, but it takes on, in signal theory, a very particular meaning. For example, we are talking about : electrical signal(telephony),electromagnetic wave(telecommunication),acoustic wave (sonar),light wave (optical fiber),binary signal (computer). We also speak of measurement signals, control signals, video signals, audio signals, etc. depending on the nature of the information transmitted.

2.2. Definition of Noise

comes from the popular Latin brugere : to bray and to roar: to roar; undesirable disturbance which is superimposed on the signal and useful data, in a transmission channel or in an information processing system and hinders the perception of this signal.

Noise corresponds to any disruptive phenomenon hindering the transmission or interpretation of a signal.

- Signal to noise ratio

The signal-to-noise ratio measures the amount of noise contained in the signal. It is expressed by the ratio of the signal powers (P_S) and noise (P_B). It is often given in decibels (dB).

$$RSB = \frac{P_S}{P_N} \quad ; \quad RSB_{dB} = 10log\frac{P_S}{P_B} \tag{1.1}$$

Or log is the decimal logarithm.

Thus, it appears obvious that a fundamental problem in signal processing will be to extract the useful signal from the noise. The difficulty of the problem depends in particular on the proportion between signal and noise. This is measured by the signal-to-noise ratio (RSB, Or SNR in English for Signal to Noise Ratio).

The SNR therefore measures the quality of the signal. It is an objective measure. However, in many cases, especially those where the human operator is involved in the processing chain, this measurement is not very meaningful. This is especially true for audio signals or images and videos. Subjective measurements, or finer measurements, taking into account the properties of human perception must be implemented.

3. Main functions of signal processing

The main functions of signal processing are:

· Analysis : We seek to isolate the essential components of a signal of complex shape, in order to better understand its nature and origins.

· The measure : Measuring a signal, in particular a random one, means trying to estimate the value of a characteristic quantity associated with it with a certain degree of confidence.

· Filtering : it is a function which consists of eliminating certain unwanted components of the signal.

· Regeneration : it is an operation by which we attempt to restore its initial form to a signal which has undergone various distortions.

· Detection : through this operation we attempt to extract a useful signal from the background noise superimposed on it.

· Identification : it is an often complementary process which makes it possible to classify the observed signal.

· Synthesis : reverse operation of the analysis, consists of creating a signal of appropriate form by proceeding, for example, with a combination of elementary signals.

· Coding : in addition to its function of translation into digital language, is used either to combat background noise or to try to save bandwidth or computer memory.

· Modulation and frequency change : are essentially means for adapting a signal to the frequency characteristics of a transmission channel, an analysis filter or a recording report.

4. Areas of application

Telecommunications, Telephony, Radar, Sonar, Image processing, Astronomy, Geophysics, Automation, Measurement technology, Study of mechanical vibrations, Monitoring of industrial processes, Acoustics, Shape recognition, Biomedical analyses, etc.

In telecommunications : whether in the field of telephony or in the transfer of digital data on land or via satellite, data compression is essential to make the best use of the available bandwidth and minimize losses. Another area of application is echo suppression.

In audio: we seek to improve recording and compression techniques to obtain the highest possible sound quality. Echo correction techniques help reduce the effects of acoustic reflections in the room. Sound processing has greatly improved thanks to computers. Sound synthesis also makes it possible to create artificial sounds or recreate the sounds of natural instruments. She was at the origin of many upheavals in music.

The analysis of echoes makes it possible to obtain information on the medium on which the waves are reflected. This technique is used in the field of radar or sonar imaging. In geophysics, by analyzing the reflections of acoustic waves, we can determine the thickness and

the nature of the subsoil strata. This technique is used in the field of mineral prospecting and in the prediction of earthquakes.

In imaging : there are applications in the medical field (tomographic reconstruction, magnetic resonance imaging - MRI), in space (processing of satellite photos or radar images). This field also includes pattern recognition and compression techniques.

The processing of video sequences concerns compression, restoration, creation of special effects, extraction of descriptors (recognition of shapes and textures, movement tracking, characterization etc.) in order to produce automatic annotations from a basic perspective data (search by content).

We can also cite some areas of application of signal processing : Measurement techniques, Study of mechanical vibrations, Monitoring of industrial processes, Pattern recognition, Biomedical analyses, Geophysics, Seismology, Astronomy, Radar, Sonar, Acoustics, etc.

5. Models and measurement of signals

5.1. Mathematical model

The mathematical model of a signal is a function of one, two or three variables : x(t) ; x(i,j); x(i,j,t). The first case is the most common: the variable t is usually time but it can also represent another quantity (a distance for example). The function represents the evolution of an electrical quantity or translated into this form by an appropriate sensor: microphone: acoustic signal, camera: video signal, etc.

5.2. Functional model

A correspondence rule between a set of real or complex numbers is called functional. In other words, a functional is a function of functions. The signals resulting from a treatment or some of their parameters are often expressed by functional relationships. For example :

Weighted integral value (weighting function g(t))

$$f_1(x) = \int_{-\infty}^{+\infty} x(t).g(t)dt \tag{1.2}$$

Weighted quadratic integral value

$$f_2(x) = \int_{-\infty}^{+\infty} x^2(t).g(t)dt \tag{1.3}$$

Convolution product

$$x(t) * g(t) = \int_{-\infty}^{+\infty} x(\tau)g(t-\tau)d\tau \tag{1.4}$$

Scalar product (evaluated on the interval T)

$$< x, y^* >= \int_T x(t).y^*(t)dt \tag{1.5}$$

Fourier transform

$$X(f) = \int_{-\infty}^{+\infty} x(t).e^{-j2\pi ft}dt \tag{1.6}$$

6. Signal classifications

We recall that a signal is a function depending on one or more variables. For example, let the signal be : s(t), s is a quantity depending on a parameter t (by convention, we will use the letter t for the time variable).

A signal can be classified according to different criteria: its dimensionality, its temporal characteristics, the values it can take, its predictability.

6.1. Dimensionality

We can take this criterion into account in two different ways: the dimension of the signal and the dimensions of the signal variables.

Let us first consider this classification criterion as being the dimension of the space of values taken by the signal (or the mathematical function modeling the signal). We then distinguish:

– the scalar signal, or single-channel signal that can take real or complex values.

– the vector signal, or multi-channel signal that can take real or complex values.

Take for example a Television (TV) signal. If we are interested in the three colors constituting an image, this TV signal takes values in a three-dimensional space, a first for red, a second for green and finally a third for blue;

[R; V; B] = TV (t).

On the other hand, if we now look at luminance, this signal takes its values in a one dimensional space; [L] = TV(t).

We can also consider this classification criterion as the dimension of the domain of the signal function, that is to say, the number of arguments taken by this function. We then distinguish:

– The one-dimensional signal which corresponds to functions with a single argument, as by

example time.

– The multi-dimensional signal which corresponds to functions with several arguments.

The TV signal corresponding to the luminance can be a function of time but also of Cartesian variables corresponding to a point on the screen; [I] = TV (t; x; y). This is a three-dimensional signal.

The signals covered in this course will be mono-dimensional depending on a variable which we will consider as time. All signal processing techniques generalize quite well to vector and multidimensional signals (see the image processing course).

6.2. Temporal characteristics

We assume a scalar signal s(t). We then distinguish:

- Continuous time signals or analog signals. The variable t $\in R$. We will note a signal analog as follows : sa(t)

- Discrete time signals: these signals are defined for certain values of the variable t.

We can represent a discrete time signal by an indexed sequence of the variable t:

$$t_n, n \ldots 0,-2,-1,0,1,2,\ldots.. \quad (1.7)$$

t_n specifies a time for which the signal is defined. Please note, this does not mean that the signal is zero between two moments; it's just not defined.

We will be interested here in a uniform distribution of moments t_n which can be noted t_n=nT where T is the time space between two consecutive samples. We can then uses s(n) Or s_n as a simplified notation. We then have the following relationships:

$$s_n = s(n) = s_a(t_n) = s_a(nT) \quad (1.8)$$

6.3. Values taken by the signal

We assume a scalar signal s(t). We then distinguish:

– Continuous-valued signals that can take a real value in a continuous interval (for example, a voltage or an electric current).

– Discrete-valued signals taking only values from a finite set of possible values.

A digital signal is a discrete time signal with discrete values. The operation of discretizing the continuous values of a signal into discrete values is a quantification, denoted q subsequently.

For example, an Analog/Digital converter processing 8-bit words; a signal quantized by this converter will take a discrete value among 256 possible.

6.4. Predictability of signals

We can distinguish two main classes of signals according to their predictability character.

– Deterministic signals which can be represented explicitly by a mathematical function.

– Random signals that evolve over time in an unpredictable manner. However, it is possible to mathematically describe certain statistical characteristics of these signals.

In this course we will mainly focus on deterministic signals.

6.5. Physically realizable signals

An experimental signal is the image of a physical process and, for this reason, must be physically realizable. It is thus subject to a whole series of constraints:

- Its energy can only be limited,

- Its amplitude is a continuous function,

- The spectrum of the signal is also necessarily limited and must tend towards zero when the frequency tends towards infinity.

7. Classification methods

Different methods of classifying signals can be considered :

- A phenomenological classification of form y=g(t) where the free variable is time;

- A spectral classification of shape Y=G(f) where the free variable is frequency;

- An energy classification: a fundamental distinction can be made between two categories of signals: Finite energy signals;

Signals with non-zero finite average power;

- A morphological classification; according to the continuous or discrete nature of the amplitude and the free variable;

- Dimensional classification; We consider one-dimensional signals S(t), the signals two-dimensional -or image- S(x,y), or even three-dimensional signals S(x,y,t) representative for example the evolution of an image as a function of time.

7.1. Phenomenological classification

This highlights the type of evolution of the signal. It may be of a predetermined nature or it may have unpredictable behavior.

A deterministic signal is a signal whose evolution over time can be perfectly predicted by an appropriate mathematical model.

On the contrary, most signals of physical origin have a non-reproducible character. Signals carrying information (speech signals, image signals, etc.) present a certain unpredictability, they will be modeled by random signals.

7.2. Energy classification

We distinguish here between signals satisfying a finite energy condition and those presenting a finite average power and infinite energy.

We will call the total energy of a signal x(t) the amount :

$$E = \int_{-\infty}^{+\infty} |x^2| \, dt \tag{1.9}$$

and average power over all times of x(t) the amount :

$$P_x = T \xrightarrow{lim} \infty \quad \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x^2| \, dt \tag{1.10}$$

The first category includes signals of transient type, whether deterministic or random (example a square or Gaussian pulse) and the second category includes signals of permanent, periodic, deterministic type and permanent random signals.

7.3. Morphological classification

Depending on whether the signal x(t) where the variable t is continuous or discrete (t_k=kT) we distinguish four types of signals (figure 1.2):

• The continuous signal in amplitude and time commonly called analog signal.

• The signal with discrete amplitude and continuous time called quantified signal.

• The continuous amplitude and discrete time signal called sampled signal.

• The discrete signal in amplitude and time called digital signal.

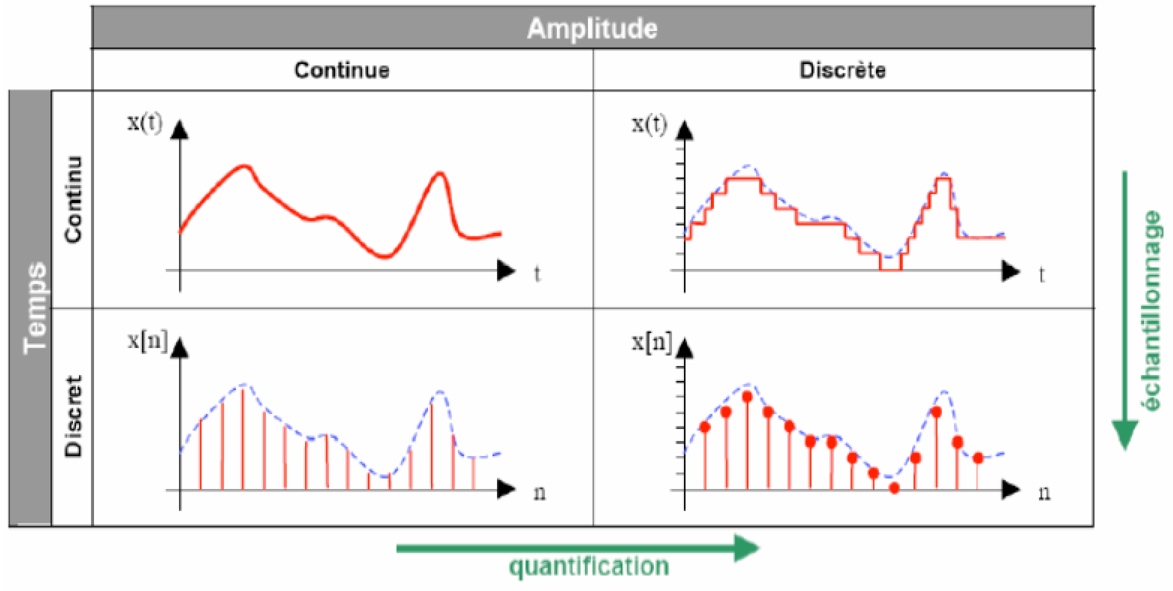


Figure 1.2. Morphological classification.

7.4. Spectral classification

The spectral analysis of a signal (or the energy distribution as a function of frequency) leads to a classification:

• Low frequency signals.

• High frequency signals.

• Narrowband signals.

• Broadband signals.

The bandwidth B of a signal is the main frequency range occupied by its spectrum. It is defined by the relation: f_2-f_1 with $0 \leq f_1 < f_2$, Or f_1 And f_2 are characteristic frequencies denoting respectively the lower and upper limits taken into account.

A signal with zero spectrum outside a specified frequency band B is called band limited or bounded-support spectrum signal. We also distinguish:

• Finite duration signals

Signals whose amplitude vanishes outside a time interval T, $x(t) = 0$ For $t \notin T$ are called signals of limited duration or limited support.

• Signals limited in amplitude

This is the case for all physically realizable signals for which the amplitude cannot exceed a certain limit value, often imposed by electronic processing devices.

• Even and odd signals

A signal is even if $x(t) = x(-t)$; it is odd if : $x(t) = -x(-t)$. Which implies that any real signal can be decomposed into an even part and an odd par t: $x(t) = x_p(t) + x_i(t)$.

• Causal signals

A signal is said to be causal if it is zero for any negative value of time : $x(t) = 0$ for $t<0$.

• **Duration of a signal**

A signal is of finite duration if it is zero outside a certain interval: $x(t) = 0, t \notin T$. These signals are called signals of limited duration or with limited support.

7.5. Dimensional classification

We consider one-dimensional signals S(t), two-dimensional signals -or image- S(x,y) , even three-dimensional signals S(x,y,t) representing for example the evolution of an image depending on time.

8. Phenomenological classification

The first classification (figure 1.3) is obtained by considering the profound nature of the evolution as a function of time. We can already classify them into two main categories: deterministic signals and random signals.

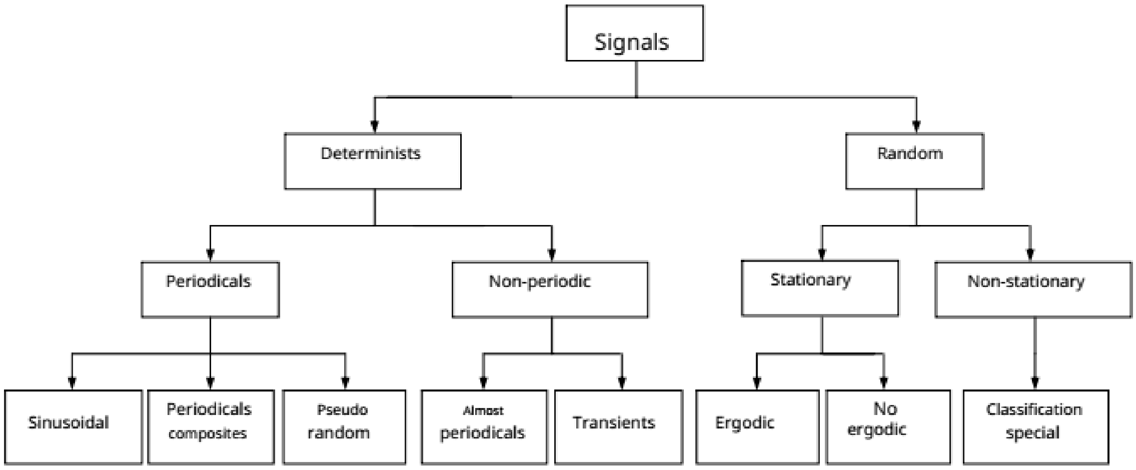


Figure 1.3. Phenomenological classification

8.1. Deterministic signals

They are signals whose evolution, as a function of the independent variable, can be perfectly predicted by an appropriate mathematical representation. They are classified as follows:

- Periodic signals, satisfying the relation:

$$S(t)=S(t+kT)\ \forall\ t \text{ And } T \text{ constant (T: called period and k:entire)} \qquad (1.11)$$

Which obey a cyclical period distribution lawT;

- Non-periodic signals, which do not benefit from this property. In the class of periodic signals, we find:

- Sinusoidal signals (figure 1.4) of general equation:

$$S(t)=A \sin(2\pi ft+\phi) \qquad (1.12)$$

Which form the most familiar group of periodic signals;

- Composite periodic signals (figure 1.3), with general equation :

$$S(t)=\sum_n [a_n \sin(n2\pi ft)+b_n \cos(n2\pi ft)] \qquad (1.13)$$

- Pseudo-random signals are periodic signals; but over a period of time, they behave like random signals. (figure 1.5). In the second class, that of non-periodic signals, we find:

- Quasi-periodic signals (figure 1.6), with general equation:

$$S(t)=\sum_i \sin(\omega_i t+\varphi_i) \qquad (1.14)$$

Which result from a sum of sinusoids of incommensurable periods;

- Transient signals (figure 1.7), of general equation:

$$S(t) = e^{-at}\sin(\omega t) \qquad (1.15)$$

Which are defined only on an interval (signals with limited support).

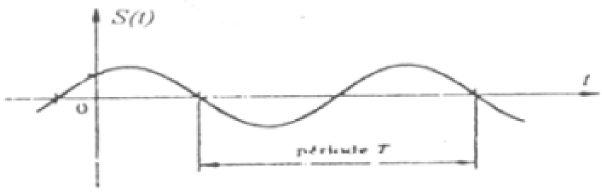


Figure 1.3. Sinusoidal signal

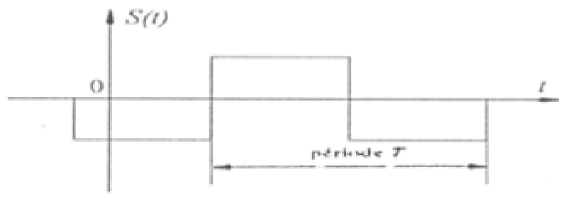


Figure 1.4. Composite periodic signal

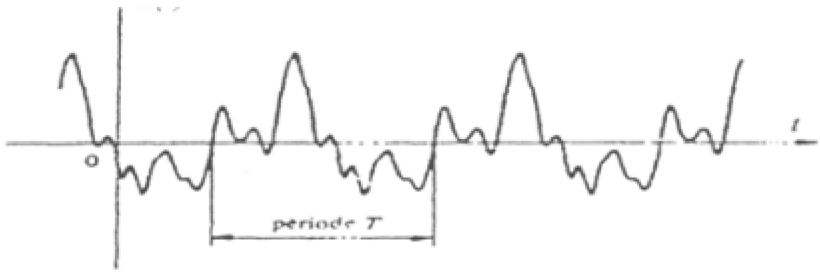


Figure 1.5. Pseudo-random signal

Figure 1.6. Quasi-periodic signal

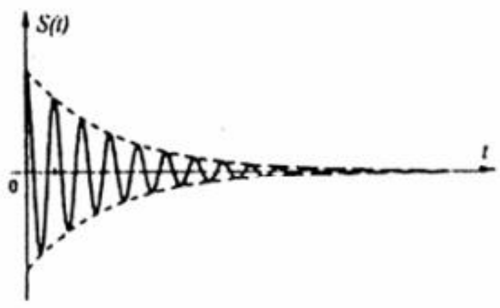


Figure 1.7. Transient signal

8.2.Random signals

These are signals whose mathematical model is not known, so their evolution over time is unpredictable. Their description is subject to statistical observations.

They obey the law of chance, we classify them as follows:

- stationary random signals, whose statistical characteristics are invariant over time; (figure 1.8)
- non-stationary random signals, which do not benefit from these properties; (figure 1.9)

In the class of stationary random signals, we find:

- Ergodic signals, if the statistical and temporal averages are identical;
- Non-ergodic signals, which do not benefit from this property.

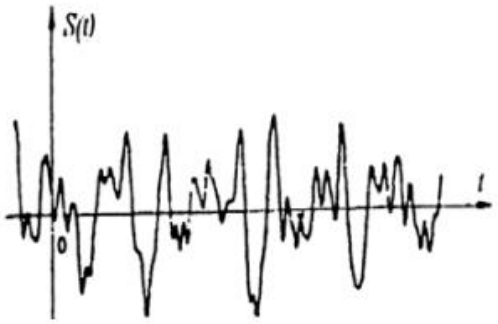


Figure 1.8. Stationary random signal in a large band.

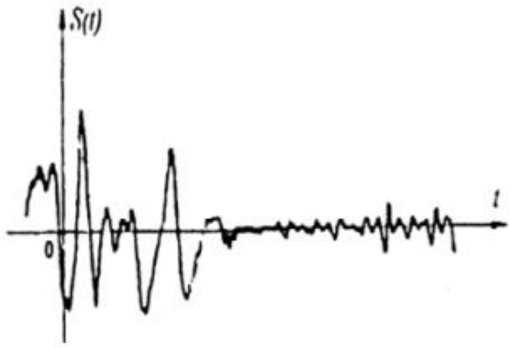


Figure 1.9. Non-stationary random signal.

9. Random process

A random process is a parameterized family of random variables, the result of the test is not a number, but a random function of time, in the case of several parameters, we speak of a random field. But we will generally limit ourselves to the case where a single variable S is sufficient. Two classes of processes are distinguished:

- Discrete process, in which the possible values of the random variable are discrete.
- Continuous process, in which the possible values of the random variable are continuous.

The aim of stochastic processes is to introduce the description of random signals carrying information to be transmitted or representing noise generated by a physical phenomenon and essentially devoted to second order properties which are generally described by the correlation function which will be described by the following.

9.1. Features

Consider k moments $t_1,...,t_k$. We can define k random variables $S(t_1),...,S(t_k)$ whose statistical description passes for the probability law (distribution function).

$$F(S_1,...,S_k;t_1,...,t_k) = prob[S(t_1) \leq S_1,...,S(t_k) \leq S_k;t_1,...,t_k] \quad (1.16)$$

Or $S_1,...,S_k$ are the states taken for the variable S(t) at time $t_1,...,t_k$ this is the order statistics k.

9.1.1. The distribution function

The distribution function $F(S_i;t_i)$ is the probability of obtaining by taking a sample from chance, a value less than or equal to $(S_i;t_i)$.

The distribution function is expressed by:

$$F(S_i;t_i) = prob\,[S(t_i) \leq S_i;t_i] \quad (1.17)$$

For the case of second-order statistics which uses a pair of random variables $S(t_1),S(t_2)$

$$F(S_1;t_1) = prob\,[S(t_1) \leq S_1;t_1] \quad (1.18)$$

For the case of order statistics1, is:

$$F(S_1,S_2;t_1,t_2) = prob\,[S(t_1) \leq S_1,S(t_2) \leq S_2;t_1,t_2]. \quad (1.19)$$

For the case of second-order statistics which uses a pair of random variables $S(t_1),S(t_2)$

9.1.2. Probability density

The probability density of the signal S(t) is by definition the derivative of the function of distribution

$$P_S(S_i;t_1) = \frac{\partial F_S(S_i;t_1)}{\partial S} \quad (1.20)$$

9.1.3. Expectation

Mathematical or average expectation $E\,[S(t_1)]$of a random variable S(t) just now t_1:

$$E[S(t_1)] = \overline{S(t_1)} = m(t_1) = \int_{-\infty}^{+\infty} S_1 P(S_1;t_1)dS_1 \quad (1.21)$$

This definition is equivalent to that of the mean of order 1 and degree 1.

9.1.4. The averages

The means of one or more random variables are defined as the mathematical expectations of the different powers of these random variables. These are statistical and temporal averages.

- Statistical averages (statistical moments):

The statistical mean of the random variable S(t) at a moment t_1, is defined by:

$$\overline{S(t_1)} = m(t_1) = \lim_{N\to\infty} \frac{S_1(t_1) + S_2(t_1) + ... + S_N(t_1)}{N} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} S_i(t_1) \quad (1.22)$$

It characterizes the position of the results distribution curve. Now consider the pair of random variables $S(t_1)$ And $S(t_2)$.

The moment of order 2 of the variables $S(t_1)$ And $S(t_2)$ is called the autocorrelation function statistics (FAC) of the process S(t) denoted by Γ_{SS}:

$$\Gamma_{SS}(t_1, t_2) = E[S(t_1).S(t_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S_1 S_2 P(S_1, S_2; t_1, t_2) dS_1 dS_2 \quad (1.23)$$

In the case of two random processes S(t),Y(t) , the moment of order 2 and degree 1 is the statistical inter-correlation function denoted by Γ_{SY}.

$$\Gamma_{SY}(t_1, t_2) = E(t_1, t_2) = E[S(t_1).Y(t_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S_1 Y_2 P(S_1, Y_2; t_1, t_2) dS_1 dY_2 \quad (1.24)$$

Noticed : The order average n, of a random variable S(t) assumed to be continuous, at an instant t_1, defined by:

$$m_n(t_1) = \overline{S^n(t_1)} = \int_{-\infty}^{+\infty} S_1^n P(S_1; t_1) dS_1 \quad (1.25)$$

- Time average (time point)

The temporal average $\overline{S_i(t)}$ of a sample $S_i(t)$ is given by the relation:

$$\overline{S_i(t)} = \overline{S(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} S(t)dt \quad (1.26)$$

The FAC, which is the time average value of the product of S(t) by S(t+ τ) given by:

$$\overline{S(t).S(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} S(t).S(t+\tau)dt \quad (1.27)$$

In the case of two random processes S(t) And Y(t), the intercorrelation function (cross-correlation) is given by the relation:

$$\overline{S(t).Y(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} S(t).Y(t+\tau)dt \quad (1.28)$$

T : Time difference or delay.

Noticed : Two signals having different amplitudes can have identical time averages in the same time interval.

9.2. Complex random process

A random process S(t) with complex values is defined in the same way as the real random process that we considered previously, but in two dimensions (R(t),I(t)) as following :

$$S(t) = R(t) + jI(t) \quad (1.29)$$

For the moment, we obtain from the first order:

$$E[S(t)] = E[R(t)] + jE[I(t)] \quad (1.30)$$

For the second order moment :

$$\forall\, t_1, t_2 :\ \Gamma_{SS}(t_1,t_2) = \Gamma_{SS}(t_2,t_1) = E[S(t_1)S(t_2)] \quad (1.31)$$

And

$$\forall\, t_1, t_2 :\ \Gamma_{SS}(t_1,t_2) = E[S(t_1)S^*(t_2)] = \Gamma^*_{SS}(t_2,t_1) \quad (1.32)$$

Or

S^* (t_2): is the conjugate of S(t_2),

Γ^*_{SS}(t1,t2): is the conjugate of Γ_{SS}(t1,t2) .

The relation (1.32) corresponds to the Hermitian symmetry $\Gamma_{SS}(t_1,t_2)$ Who is called the function statistical auto-correlation of the complex-valued random process.

Likewise, the definitions of temporal moments relate to complex processes. Thus the auto-correlation function becomes:

$$\overline{S^*(t)S(t+\tau)} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} S^*(t)S(t+\tau)dt \qquad (1.33)$$

9.3.Stationarity

The notion of stationarity plays an essential role in the study of random signals, we find:

9.3.1. Strictly stationary process (in the strict sense)

A random signal S(t_2)is strictly stationary if its temporal law is invariant any change in the origin of time can be expressed by:

$$F(S_1,...,S_k ; t_1,...,t_k) = F(S_1,...,S_k ; t_{1+\tau},..., t_{k+\tau}) \qquad (1.34)$$

For everything k, all τ and all series of moments $t_1,...,t_k$.

9.3.2. Weakly stationary process (in the broad sense)

A weakly stationary or second-order stationary signal if only its probabilistic characteristics of order 1 and order 2 are invariant for any change in the origin of times.

In this case we have:

- E[S(t)] = Cst (Independent of t).
- $\Gamma_{SS}(t_1,t_2) = E[S^*(t_1)S(t_2)] = E[S^*(t_1)S(t_{1+\tau})] = \Gamma_{SS}(\tau)$ (Function uniquely of $\tau = t_2 - t_1$).
- $\Gamma_{SS}(\tau)$ is continuous at the origin.

Noticed :

- A second order stationary random signal is characterized by its correlation function

- The correlation function of a real signal is even since according to second order stationarity

$$E[S(t)S(t-\tau)] = E[S(t+\tau)S(t)]$$

From where :

$$\Gamma_{SS}(\tau) = \Gamma_{SS}(-\tau) \quad (1.35)$$

9.4. Ergodicity

A process is said to be ergodic when all its temporal averages (temporal moments) are identical, we write :

$$\overline{S_1(t)} = \overline{S_2(t)} = \cdots = \overline{S_N(t)} \quad (1.36)$$

If the process is stationary and ergodic:

$$E[S(t)] = \overline{S(t)} \quad (1.37)$$

Noticed : Ergodicity and stationarity are independent.

10. Signals and Systems

A system is a physical entity that performs an operation on a signal. A system therefore defines an input signal and an output signal; the output signal corresponds to the transformation carried out by the system on the input signal. For example, the human ear is a system transforming a signal corresponding to a variation in sound pressure into parallel sequences of electrical signals on the auditory nerve. A microphone is a system somewhat similar to the previous one (in a very reductive first approximation...) insofar as a variation in acoustic pressure is transformed into a one-dimensional electrical signal. The study of such systems leads to analyzing the transformations between input and output signals for more or less complex systems; this activity is called signal processing. We will only talk here about digital signal processing.

11. Special functions

11.1. Dirac impulse

The Dirac momentum δ(t) (figure 1.10), also called unit impulse or delta distribution, is defined by the scalar product:

$$x(0) = \langle x, \delta \rangle = \int_{-\infty}^{+\infty} x(t)\delta(t)dt \tag{1.38}$$

in a general way :

$$x(t_0) = \int_{-\infty}^{+\infty} x(t)\delta(t - t_0)dt \tag{1.39}$$

in particular, by posing x (t) = 1, we obtain

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1 \tag{1.40}$$

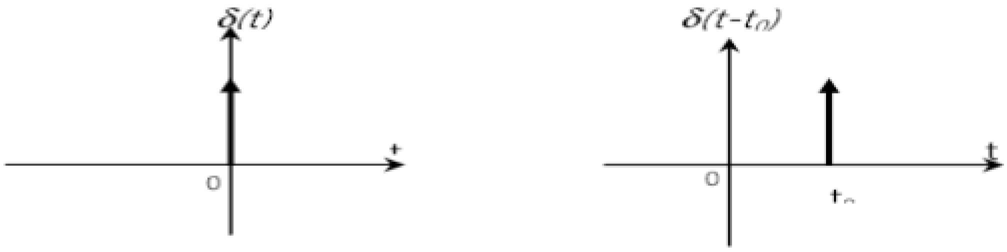


Figure 1.10. Dirac pulse

Properties :

• Either x(t) a continuous function in t = 0 or t = t_0

$x(t).\delta(t) = x(0).\delta(t)$

$x(t).\delta(t - t_0) = x(t_0).\delta(t - t_0)$

• Identify :

$x(t) * \delta(t) = x(t)$

• Translation :

$x(t) * \delta(t - t_0) = x(t - t_0)$

$x(t - t_1) * \delta(t - t_2) = x(t - t_1 - t_2)$

$\delta(t - t_1) * \delta(t - t_2) = \delta(t - t_1 - t_2)$

• Change of variable:

$\delta(at) = |a|^{-1}\delta(t)$

$\delta(w) = \dfrac{1}{2\pi}\delta(f)$

• Dirac periodic pulse sequence: (Dirac comb – figure 1.11)

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$x(t).\delta_T(t) = \sum_{k=-\infty}^{+\infty} x(kT).\delta(t - kT)$$

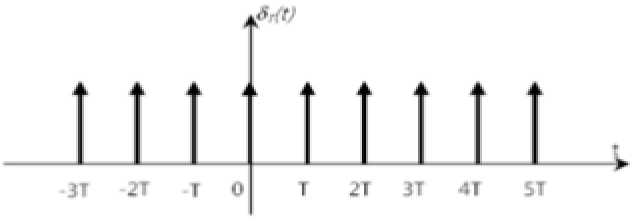


Figure 1.11. Dirac's comb

$$sgn(t) = \begin{cases} -1 & t < 0 \\ +1 & t > 0 \end{cases} \quad = \frac{t}{|t|} \; pour \; t \neq 0$$

By convention the original value is zero.

11.3. Unit jump function (or Echelon)

$$\epsilon(t) = \frac{1}{2} + \frac{1}{2} sgn(t) = \begin{cases} 0, & t < 0 \\ 1, & 1 > 0 \end{cases}$$

The step function is not defined for t = 0

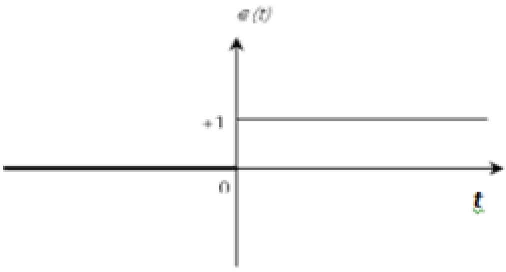


Figure 1.13. Step function

11.4. Ramp function

The ramp function can be defined from the unit jump function :

$$r(t) = \int_{-\infty}^{t} \epsilon(\tau)\, d\tau = t.\epsilon(t) \;\Rightarrow\; \epsilon(t) = \frac{dr(t)}{dt} \; pour \; t \neq 0$$

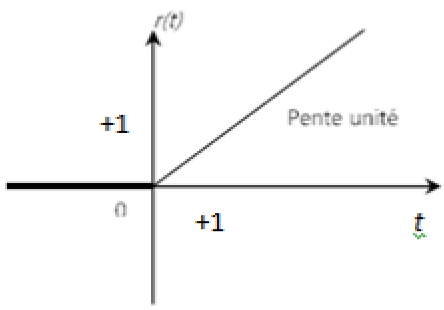


Figure 1.14. Ramp function

11.5. Door function

$$rect(t) = \epsilon\left(t + \tfrac{1}{2}\right) - \epsilon\left(t - \tfrac{1}{2}\right) = \begin{cases} 1, & |t| < \tfrac{1}{2} \\ 0, & |t| > \tfrac{1}{2} \end{cases}$$

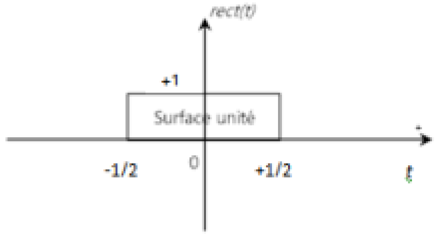
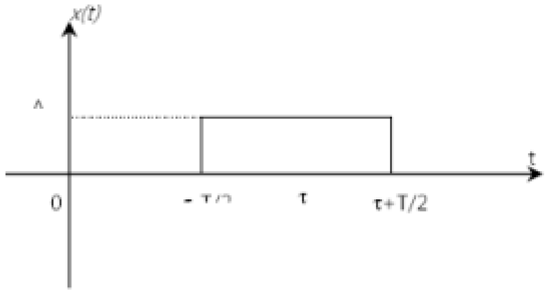


Figure 1.15. Function Door centered in t = 0.

When introducing the change: t = t/T we obtain in a more general way for a rectangular pulse of duration T centered in t = τ:

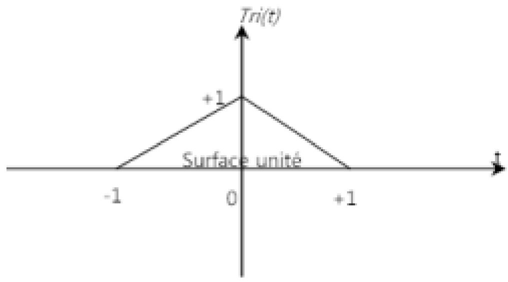
$$x(t) = A\, rect\left(\frac{t-\tau}{T}\right)$$



11.6. Triangle function

The normalized triangular function:

$$Tri(t) = \begin{cases} 1 - |t|, & |t| \leq 1 \\ 0, & |t| \geq 1 \end{cases}$$



We can also write:

$$Tri(t) = rect(t) * rect(t)$$

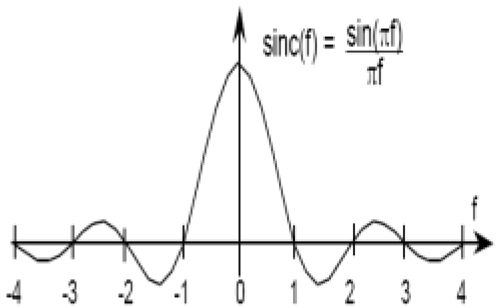
Likewise :

$$x(t) = A\, Tri\left(\frac{t-\tau}{T}\right)$$

11.7. Cardinal sine function

$$sinc(f) = \frac{sin\pi f}{\pi f}$$

$$sinc(0) = 1$$



12. Notions of power and energy

12.1. Energy of a signal

Either x(t) any signal (complex function),

- Energy on $[t_1,t_2]$ is defined by:

$$W_x(t_1,t_2) = \int_{t_1}^{t_2} |x(t)|^2\, dt \tag{1.41}$$

Where the notation $|x(t)|^2$ means x(t) * $x^*(t)$; $x^*(t)$: is x(t) conjugated

Finite energy signals satisfy the following condition :

$$E_x = \int_{-\infty}^{+\infty} |x^2(t)|\, dt < \infty$$ Where E_x is the total energy of the signal $x(t)$ (1.42)

12.2. Average signal strength

Either x(t) any signal (complex function), the average power on $[t_1,t_2]$ is defined by:

$$P_x(t_1,t_2) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} |x(t)|^2\, dt \tag{1.43}$$

Special case of periodic period signalsT_0

$$x(t) = \sum_{k=-\infty}^{k=+\infty} x_p(t-kT_0) \tag{1.44}$$

Or $x_p(t)$ is the signal over a period T_0, then the average power over a period is equal to:

$$P_x = \lim_{T\to 0}\frac{1}{T}\int_{-T}^{T} |x(t)|^2 dt = \frac{1}{T_0}\int_{-T_0}^{T_0} |x(t)|^2\, dt = \frac{1}{T_0}\int_{-T_0}^{T_0} \left|x_p(t)\right|^2 dt = P_{x_p} \tag{1.45}$$

12.3. Finite energy signals

A signal x(t) is said to have finite energy if it is summable square, that is to say if:

$$W_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty \tag{1.46}$$

Which implies $P_x=0$.

12.4. Finite average power signals

A signal x(t) is said to have finite average power if:

$$P_x = \lim_{T\to +\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 dt < \infty \tag{1.47}$$

Case of periodic signals of period T:

$$P_x = \frac{1}{T}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} |x(t)|^2 dt < \infty \tag{1.48}$$

If $P_x \neq 0$, then $W_x = \infty$ (infinite total energy signal).

Example :

Calculate in each case the total energy and the total average power (a > 0).

$Arect(t/T);\ Asinwt;\ \delta(t); Ae^{-at}; A\, tri(t/T)$

Solution :

$\boldsymbol{P = 0, E = A^2T\, ; P = \frac{A^2}{2}, E = \infty, P = \frac{1}{2}, E = \infty; P = \infty, E = \infty; P = 0, E = \frac{2A^2T}{3}}$

CHAPTER II

Fourier analysis

Introduction

The aim of this chapter is to introduce Fourier analysis in the context of linear electronic systems. This analysis is a frequency type analysis, extended to regimes which are not necessarily sinusoidal. Fourier analysis is widely used in electricity as in physics. We introduce complex and real Fourier series. The terms of the Fourier series are sinusoidal and cosine functions. Once again, we see the importance of the harmonic analysis of systems, since the relevance of these decompositions is guaranteed for any linear system (principle of superposition).

The Fourier transform has already been pointed out as a mathematical special case of the Laplace transform. It is widely used in all technical branches with vast and diverse implications: from uncertainty relationships in physics to reciprocal spaces in crystallography, including of course electricity. For this second part of the chapter, we limit ourselves to the definition of the Fourier transformation where we approach the notion of spectrum of a signal. For more comprehensive information, we advise the reader to refer to an introduction to signal processing, a field where this mathematical tool is essential.

1. Fourier series

Any periodic signal is decomposed into a sum of sinusoidal signals, this is a remarkable property.

1.1. Complex Fourier series

Function $x: t \rightarrow x(t)$; $t \in reals$, defined on the interval [t_1, t_1+T], maybe expressed as a series of functions:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{-j2\pi\frac{n}{T}t} \tag{2.1}$$

All functions:

$$\left\{\psi_n;\ \psi_n(t) = e^{j\frac{2\pi}{T}t} = \cos\left(\frac{2\pi n}{T}t\right) + jsin(\frac{2\pi n}{T}t)\right\} \qquad (2.2)$$

Constitutes a basis of the vector space containing the function x, and the coefficients X_n constitute the projections of the function x on this basis.

We use the usual scalar product and we obtain, for the calculation of these coefficients :

$$X_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-j2\pi\frac{n}{T}t}dt \qquad (2.3)$$

1.2. Frequency spectrum

The different frequencies of the Fourier series decomposition are given by :

$$f_n = \frac{n}{T}, \qquad n = 0,1,2,\ldots \qquad (2.4)$$

The frequency spectrum is given by the graph :

$$\{(f, X_n)\} \qquad (2.5)$$

or physically: the amplitudes associated with the different frequencies.

This frequency spectrum is therefore a way of representing a periodic signal, and this remains valid in the general case of a non-periodic signal (of finite energy), which we will see with the Fourier transform.

The frequency spectrum is discrete here, it contains:

- the continuous level : average signal value

- the fundamental component, of the signal frequency

- harmonics, of frequencies multiples of that of the fundamental

- negative frequencies,which have no direct physical significance; we owe their presence mathematically to the development of the real function in complex series.

These negative frequencies disappear with the use of real Fourier series.

1.3. Example: decomposition of a train of pulses

The following pulse is decomposed into a complex Fourier series, by choosing a period T :

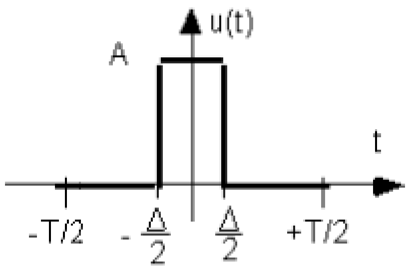


Figure 2.1. Example : decomposition of a train of pulses

All calculations carried out we obtain for the coefficients:

$$X_n = \frac{A}{\pi n}\sin\left(\frac{\pi n \Delta}{T}\right) \tag{2.6}$$

Taking the discrete frequency as a variable :

$$f_n = \frac{n}{T}, \qquad n = 0,1,2,\ldots \tag{2.7}$$

We obtain the following expression :

$$X_n(f_n) = \frac{A\Delta}{T}\frac{\sin(\pi\Delta f_n)}{\pi\Delta f_n} \quad \text{(Shape envelope sin x/x)} \tag{2.8}$$

We obtain, for the representation of the spectrum of this impulse:

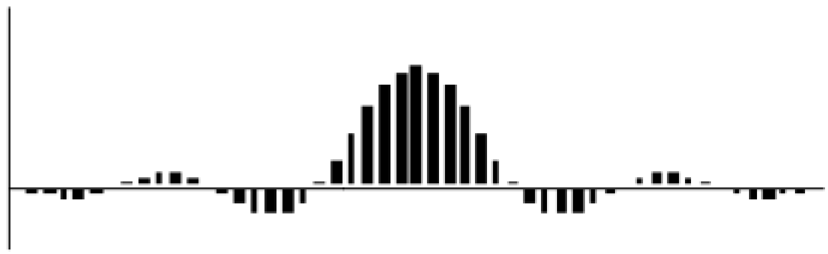


Figure 2.2. Example: Discrete frequency spectrum of the pulse.

It should be noted that if we examine the sum of the Fourier series over the entire time axis, we obtain a periodic signal :

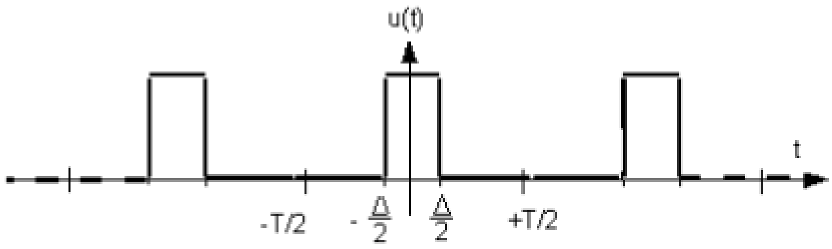


Figure 2.3. Example: pulse train

It therefore has two possible approaches: either we are only interested in a portion of the signal (pulse over a time interval T) and then the series only takes on meaning over this interval, or we develop over the entire axis real a periodic signal thanks to this Fourier decomposition. It is the latter case that is of general interest, because non-periodic signals are processed using the Fourier transform which generates a continuous spectrum.

1.4. Real Fourier series

As the electrical signal is represented by a real function with real values, we can also treat this case without going through complex numbers.

We have the following development, for the real Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\left(2\pi \frac{n}{T} t\right) + b_n \sin(2\pi \frac{n}{T} t)) \tag{2.9}$$

with, for the coefficients:

$$a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt$$

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos(2\pi \frac{n}{T} t)dt$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin(2\pi \frac{n}{T} t)dt$$

Odd signals develop into a sine series, and even signals develop into a cosine series, which further simplifies the calculations. The spectrum obtained is unilateral, hence the name unilateral Fourier series.

In the previous example of the train of rectangular pulses (figure 2.3). We obtain, as a one-sided Fourier expansion:

$$x(t) = \frac{A\Delta}{T} + 2\frac{A\Delta}{T}\sum_{n=1}^{\infty}\frac{\sin(\pi\Delta\frac{n}{T})}{\pi\Delta\frac{n}{T}} \tag{2.10}$$

And for the graphical representation of the discrete (unilateral) spectrum :

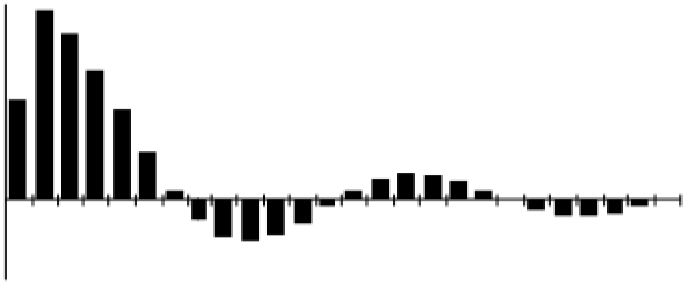
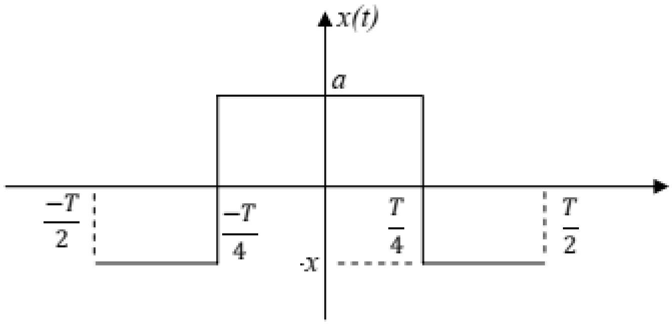


Figure 2.4. Example: Discrete unilateral frequency spectrum of the pulse.

Note that the unilateral spectrum is not the truncated version of the bilateral spectrum: the harmonics have double the amplitude compared to the latter, except in the special case, that of zero frequency. It should be seen that the bilateral spectrum of a sinusoidal signal is given by the two frequencies: the positive and the negative, and their amplitude is half that of the frequency of the unilateral spectrum.

Example :

Or an even signal x(t) of period T defined on [0,T/2[represented by the figure above:



- Develop the periodic signal in Fourier series x(t).

The following development in Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(2\pi \frac{n}{T} t\right)$$

with, for the coefficients:

$a_0 = 0$; $a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos(2\pi \frac{n}{T} t)dt$; $b_n = 0$: signal pair

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(nwt)dt = \frac{4a}{nwT}\left([\sin(nwt)]_0^{\frac{T}{4}} - [\sin(nwt)]_{\frac{T}{4}}^{\frac{T}{2}}\right) = \frac{4a}{n\pi}\sin(\frac{n\pi}{2})$$

Pour n pair $\sin\left(\frac{n\pi}{2}\right) = 0$, pour $n = 2k + 1$, alors $\sin\left(\frac{n\pi}{2}\right) = (-1)^k$, d'où :

$a_k = \frac{4a}{\pi}\frac{(-1)^k}{2k+1}$ et $\boldsymbol{x(t) = \frac{4a}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos((2k+1)\,wt)}$

1.5. Alternative Fourier series

It is defined by:

$$x(t) = A_0 + \sum_{n=1}^{\infty}(A_n\cos(2\pi n f_0 t - \varphi_n)), \qquad A_0 = a_0 \tag{2.11}$$

$$A_n = \sqrt{{a_n}^2 + {b_n}^2} \qquad \varphi_n = Arctg\,\frac{b_n}{a_n}$$

A_n: amplitude of the spectral component; φ_n: phase of the spectral component.

1.6. Development of a periodic function in Fourier series

1.6.1. Imaginary exponential series

• Fourier's theorem:

Either : $f: R \to C$ periodic of period T, pulsation $k = \frac{2\pi}{T}$ If f is summable square on [0,T],so

$f(x) = \sum_{n=-\infty}^{n=+\infty} C_n\, e^{ik_n x}$ or for $n \in Z$:

$k_n = n \times \frac{2\pi}{T}$ et $C_n = \frac{1}{T}\int_{x_0}^{x_0+T} f(x)e^{-ik_n x}dx$

• Fourier spectrum: It is $\{|C_n|, n \in Z\}$: (In general, $|C_n|$ decreases when $|n|$increase)

1.6.2. Sine and cosine series

• General case :

$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(k_n t) + b_n\sin(k_n t))$

$a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt$; $a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos(k_n t)dt$; $b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin(k_n t)dt$

• Parity:

If x(t) is even, we will have ∀ n∈N, $b_n=0$

If x(t) is odd, we will have ∀ n∈N, $a_n=0$

• Case of a real function:

If a_n, b_n∈R, $a_n \cos(k_n x) + b_n \sin(k_n x) = a'_n \cos(k_n x + \varphi_n)$

and then $x(t)=a_0 + \sum_{n=1}^{\infty} a'_n \cos(k_n t + \varphi_n)$

The term for n=1 is called the fundamental or first harmonic. The one for n=2 is called second harmonic, etc.

1.6.3. Bessel–Parseval equality

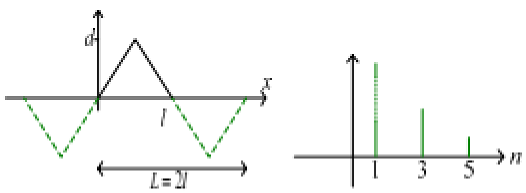
$$\frac{1}{T}\int_{t_0}^{t_0+T}|x(t)|^2dt = \sum_{n=-\infty}^{+\infty}|C_n|^2 = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{+\infty}|a_n|^2 + |b_n|^2 \quad (2.12)$$

1.7. Development of a closed support function

1.7.1. Example

We have $k_n = 2n\frac{\pi}{L} = n\frac{\pi}{l}$, and the function is odd. We then find:

$$x(t) = \frac{8d}{\pi^2}(sin\frac{\pi t}{l} - \frac{1}{9}sin\frac{3\pi t}{l} + \frac{1}{25}sin\frac{5\pi t}{l} + \cdots$$



1.7.2. Spatial and temporal function

- Spatial functions

x: abscissa, L:length (or wavelength), $k = \frac{2\pi}{L}$ spatial pulsation. We have:

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{ik_n x}$$

- Temporal functions

$x \rightarrow t$, $L \rightarrow T$, $\omega = \frac{2\pi}{T}$

ω: temporal pulsation; $f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{iw_n t}$.

2. The Fourier transformation

In electronics and signal processing, signals are not all periodic, this is even the exception. The development in Fourier series therefore does not necessarily represent the preferred analysis tool, since it is necessary for this to have periodic signals.

2.1. Fourier transformation: definition

The Fourier transformation can be seen mathematically as a special case of that of Laplace, by posing $p = j2\pi f$, for the frequency variable. We define :

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t}\, dt \qquad (2.13)$$

Or

$$x(f) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi f t}\, df \qquad (2.14)$$

Function : $X: f \to X(f)$ is the Fourier transform of the function: $x: t \to x(t)$. In signal processing, we more readily use the variable frequency f(Hz) that the pulsation

$w = 2\pi f \left[\frac{rad}{s}\right]$, usually used as a Fourier transform.

We say that x(t) And X(f) form a Fourier transform pair, it is denoted by:

$$x(t) \rightleftharpoons X(f) \qquad (2.15)$$

The Fourier transform exists if the three DIRICHLET conditions are verified (these are sufficient but not necessary conditions):

- x(t) has a finite number of discontinuities on any finite interval,

- x (t) has a finite number of maxima and minima on any finite interval,

- x(t) is absolutely integrable, i.e.:

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty \qquad (2.16)$$

It is important to note that all finite energy signals, i.e. all signals of L_2.

$$\int_{-\infty}^{+\infty} \left|x(t) e^{j2\pi f t}\right|^2 dt < +\infty \qquad (2.17)$$

admit a Fourier transform.

Example :

We notice $rect_T(t)$ the rectangular pulse defined by:

$$rect_T(t) = x(t) = \begin{cases} A, & si\ t \in [-T/2, T/2] \\ 0, & ailleurs \end{cases} \quad (2.18)$$

We then search to calculate the Fourier transform (TF) of x(t)

$$X(f) = TF\{rect_T(t)\} = A\int_{-T/2}^{T/2} e^{-j2\pi ft}\,dt = A[\frac{e^{-j2\pi ft}}{-j2\pi f}]_{-T/2}^{T/2} = A\frac{1}{j2\pi f}[e^{j\pi fT} - e^{-j\pi fT}]$$

And finally $X(f) = AT\frac{\sin(\pi fT)}{\pi fT} \equiv ATsinc(\pi fT)$: cardinal sine function.

2.2. Amplitude spectrum and phase spectrum

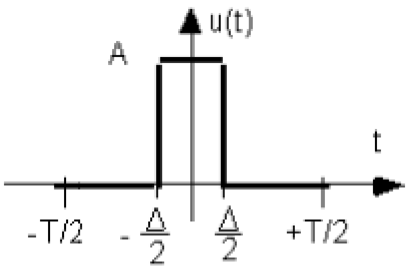
In the general case, the Fourier transform of a function produces a complex-valued function. Thus, we can obtain two pieces of information from the Fourier transformed function:

The amplitude spectrum :$\{(f, |X(f)|)\}$

The phase spectrum:$\{(f, \arg(X(f))\}$

2.3. Example :

We take the previous pulse with the Fourier transform:



Impulse equation : $x(t) = A\ rect\ (\frac{t}{\Delta})$

All calculations done, we obtain for its Fourier transform: $X(f) = A.\Delta\frac{\sin(\pi\Delta f)}{\pi\Delta f}$. We notes that in this case, X(f) is a real function. It can be represented graphically:

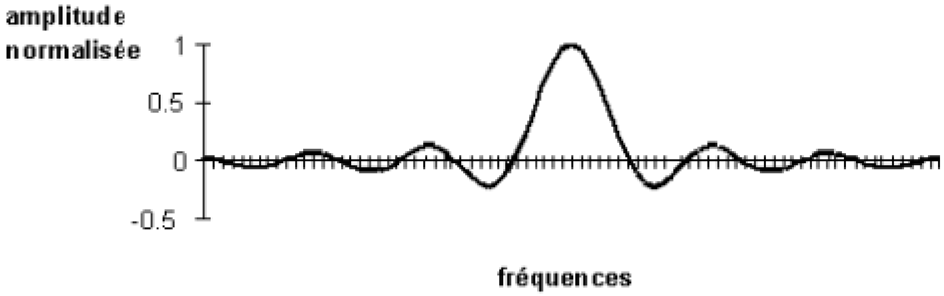


Figure 2.5. Example: Fourier transform of the rectangular signal.

As X(f) is real, its phase spectrum is zero for the positive parts of the TF only, and its amplitude spectrum has the following appearance:

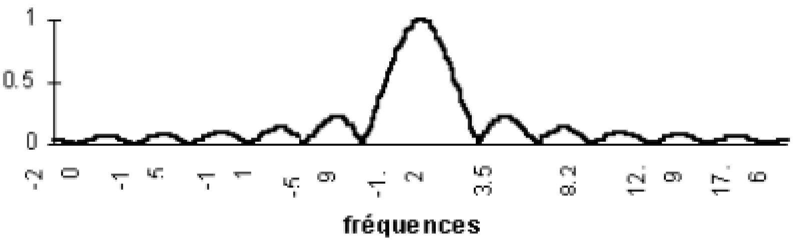


Figure 2.6. Example: Amplitude spectrum of the rectangular signal.

Remarks

As with the development in Fourier series, we witness the appearance of negative frequencies, which cannot be interpreted directly, but which nevertheless carry energy.

The Fourier transform here corresponds to the envelope of the discrete spectrum of the Fourier expansion. In this Fourier transformation, all the frequencies are used for the frequency representation of the time signal: the spectrum is continuous.

Unlike the development in Fourier series which generates a periodic function on the entire real axis whatever the values taken by this function outside the period considered, the Fourier transformation is applied to the function acting on the entire real axis . A correspondence is thus created between the temporal space where the signal evolves, and the slightly more abstract frequency space. Electricians call this time-frequency duality. Crystallographers talk about direct space and reciprocal space, etc.

As already mentioned previously, the usefulness of this transformation is to obtain another representation of a signal. This frequency representation is essential in signal processing. The situation is analogous to that prevailing for the Laplace transformation, but here the space given by the Fourier transformation is well identified: it is a space of frequencies:

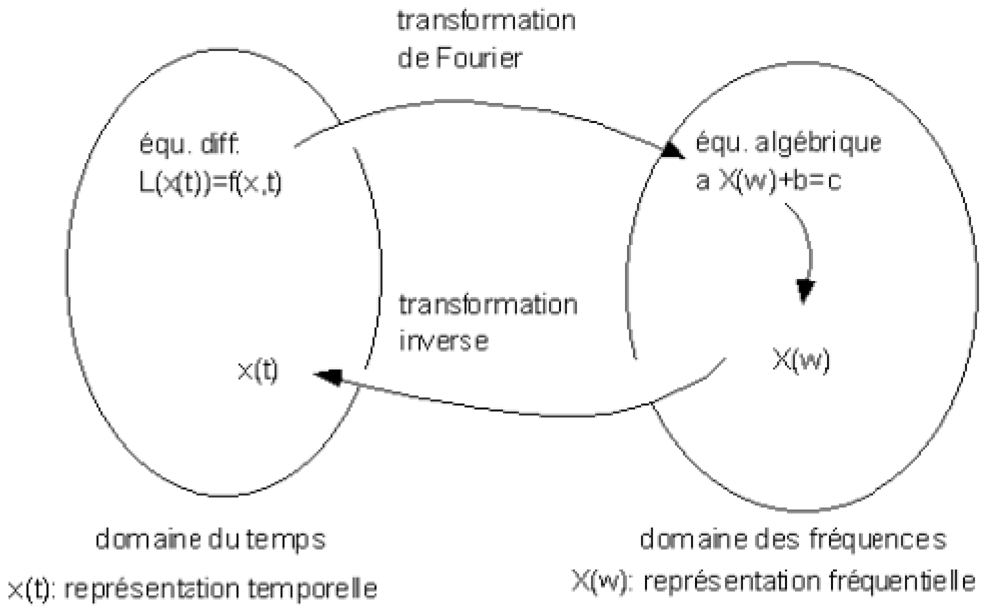


Figure 2.7. Time-frequency duality.

2.4. Transfer function

Here we present an example, where we use the Fourier transform, to solve a differential equation. This is not the main use of this tool, but it allows you to make a point regarding the transfer functions.

If we reduce the Laplace transformation to that of Fourier, we take as variable:

2.Thus, the Laplace transfer function transforms into that of Fourier with

this substitution. And this Fourier transfer function is nothing other than that obtained with complex numbers and which in fact corresponds to the transfer function in the harmonic regime.

System block diagram:

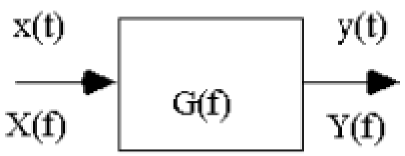


Figure 2.8. Block diagram of the system.

In temporal space, we have :

$$L(x(t)) = y(t) \tag{2.19}$$

L: linear operator; x(t): system excitation; y(t): system response.

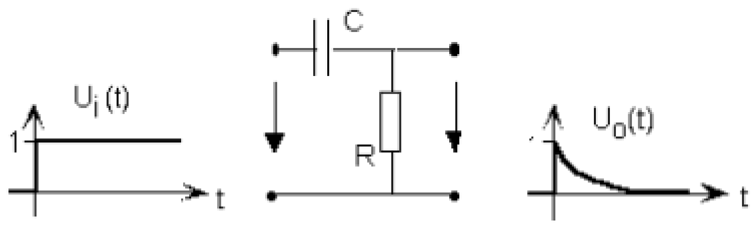
In frequency space, we obtain:

$$Y(f) = G(f)X(f) \tag{2.20}$$

X(f): Fourier transform of the excitation; Y(f): Fourier transform of the response. G(f) : Transfer function.

Example: RC cell excited by a unit step

Consider an RC cell, to which we apply a unit scale:



By the voltage divider in the field of p, we obtain the Laplace transfer function:

$$G(p) = \frac{RCp}{RCp+1}$$

Fourier transfer function:

$$G(f) = \frac{RCj2\pi f}{RCj2\pi f+1}$$

Input signal:

$$U_{in}(f) = \frac{1}{j2\pi f}$$

Output signal:

$$U_0(f) = G(f).U_i(f) = \frac{1}{j2\pi f + {}^{1}/_{RC}}$$

Inverse transformation of the output signal:

$$U_0(t) = e^{-\frac{t}{RC}}$$

2.5. Main properties of the Fourier transform

Linearity

If $x_1(t) \rightleftharpoons X_1(f)$ alors, $\forall c_1, c_2 \in \mathbb{C}$

$x_2(t) \rightleftharpoons X_2(f)$ $c_1x_1(t) + c_2x_2(t) \rightleftharpoons c_1X_1(f) + c_2X_2(f)$

Scale Property – Dilation

$$x(at) \rightleftharpoons \frac{1}{|a|_1} X\left(\frac{f}{a}\right)$$

Time delay

$$x(t - t_0) \rightleftharpoons X(f)e^{-j2\pi f t_0}$$

Frequency shift

$$e^{j2\pi t f_0}x(t) \rightleftharpoons X(f - f_0)$$

Amplitude modulation

$x(t) = Acos(2\pi f_0 t).m(t)$; $x(t)$: le signal modulé en amplitude, $m(t)$: est le message

$$X(f) = \frac{A}{2}[M(f - f_0) + M(f + f_0)]$$

Averages

$$X(0) = \int_{-\infty}^{+\infty} x(t)dt \quad ; \quad x(0) = \int_{-\infty}^{+\infty} X(f)df$$

Differentiation in the time domain

$$\frac{dx(t)}{dt} \rightleftharpoons j2\pi f X(f) \quad ; \quad \frac{d^n x(t)}{dt^n} \rightleftharpoons (j2\pi f)^n X(f)$$

Integration in the time domain

$$\int_{-\infty}^{t} x(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f} X(f)$$

Duality property

If $x(t) \rightleftharpoons X(f)$ alors, $X(t) \rightleftharpoons x(-f)$

Conjugation properties and symmetry

If $x(t) \rightleftharpoons X(f)$ alors, $x^*(t) \rightleftharpoons X^*(-f)$; $x(-t) \rightleftharpoons X(-f)$; $x^*(-t) \rightleftharpoons X^*(f)$

We deduce that, if x(t) is real, then: X(f) =X^*(-f), And :

- The real part of X(f) is even,
- The imaginary part of X(f) is odd,
- The module of X(f), |X(f)| is even,
- The phase of X(f), φ(f) is odd.

Parity

Odd : $x(t) = x(-t) \rightleftharpoons X(f) = X(-f)$

Even : $x(t) = -x(-t) \rightleftharpoons X(f) = -X(-f)$

Impulse of Dirac

$$\delta(t) = \begin{cases} 0, & si \;\; t \neq 0 \\ +\infty, & si \;\; t = 0 \end{cases} \; ; \int_{-\infty}^{+\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{+\infty} x(t)\delta(t - t_0)dt = x(t_0) \; ; TF\{\delta(t)\} = 1, \; TF\{1\} = \delta(f) = \delta(-f), \forall f$$

$$\delta(t - \tau) \rightleftharpoons e^{-j2\pi f\tau} \; ; \; e^{j2\pi f_0 t} \rightleftharpoons \delta(f - f_0)$$

2.6. Parseval's theorem

Parseval's equality, sometimes called Parseval's theorem, is a fundamental formula in the theory of Fourier series. This formula can be interpreted as a generalization of the Pythagorean theorem for series in Hilbert spaces. In many physical applications (electric current for example), this formula can be interpreted as follows: the total energy is obtained by summing the contributions of the different harmonics. The total energy of a signal does not depend on the representation chosen: frequency or time.

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df \qquad (2.21)$$

2.6.1. Energy, Effective value by the Fourier series – Paseval formula

The effective value of the signal is given by:

$$X_{eff_{(t_1,t_2)}}^2 = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^2(t)dt = {a_0}^2 + \frac{1}{2} \sum_{n=1}^{\infty} {A_n}^2 \qquad (2.22)$$

CHAPTER III

Laplace transform

Introduction

The Laplace transform is an integral operation that allows you to transform a function of a real variable into a function of a complex variable. By this transformation, a linear differential equation can be represented by an algebraic equation. It also makes it possible to represent particular functions (Heaviside distribution, Dirac distribution, etc.) in a very elegant way. It is these possibilities that make the Laplace transformation interesting and popular with engineers. This transformation gave rise to the technique of operational calculation or symbolic calculation which facilitates the resolution of linear differential equations which will represent the systems that we are going to study.

1. Laplace transform

1.1. Definition

Either f(t) a real or complex valued function of the real variable t defined from [0 to ∞ [And either $p = \alpha + j\beta$ a complex variable; the expression:

$$\mathcal{L}(f(t)) = F(p) = \int_0^\infty e^{-pt} f(t)dt \quad (3.1)$$

Where the symbol $\mathcal{L}(f(t))$ means the Laplace transform. In this case, it is called the One-sided Laplace transformation.

The Laplace transform therefore makes it possible to transform the time problem into the frequency domain. When we obtain the desired answer in the frequency domain, we transform the problem again into the time domain, using the inverse Laplace transform. In the Laplace domain, derivatives and integrals are combined using simple algebraic operations; no need for differential equations.

We divide the Laplace transform into two types:

- Functional transformation: it is the Laplace transform of a specific function, like $sinwt, t, e^{-at}$, etc.

- Operational transformation : it is a mathematical property of the Laplace transform, like the calculation of the derivative of *f* (t).

1.2. Exponential order

We will say that a function *f* (t) is of exponential order at infinity if and only if, there exists a pair of real numbers and *M* such as :

$$|f(t)| = Me^{\alpha t}, \forall t \geq 0 \tag{3.2}$$

1.3. Existence of the Laplace Transformation

Either f(t) a piecewise continuous function on the closed interval [0, a] (for everything a > 0) and having an exponential order at infinity such that $|f(t)| = Me^{\alpha t}, \forall t \geq 0$; then, the transformation of laplace $\mathcal{L}(f(t))$ exists and is defined for $p > \alpha$

1.4. Uniqueness of the Laplace Transformation

Let *f* (t), And *g*(t), two piecewise continuous functions with exponential order at infinity. Suppose that:

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}(g(t))$$

So *f* (t) = g (t) for $t \in [0, D]$, for ever $D > 0$, except perhaps at a finite number of points.

Example 1:

If *f* (t) = 1, then: $\mathcal{L}\big(f(t)\big) = \mathcal{L}(1) = \int_0^{\infty} e^{-pt}dt = \frac{1}{p}$

in this example, the integral converges if and only if the real part of p > 0

Example 2:

If *f* (t) = e^{at} so $\mathcal{L}\big(f(t)\big) = \mathcal{L}(e^{at}) = \int_0^{\infty} e^{at}e^{-pt}dt = \frac{1}{p-a}$, :

There is convergence if R_e {(p-a)}> 0 or R_e {p} > R_e {a}. Such as R_e : represents the real part.

1.4.1. Bilateral Transform

We also define a Laplace transformation on the domain Rreal numbers:

$$\mathcal{L}\big(f(t)\big) = F(p) = \int_{-\infty}^{\infty} e^{-pt}f(t)dt \tag{3.3}$$

This transformation is not used much in the field of engineering because we consider signals which respect causality and therefore which exist from an instant t_0.

1.5. Inverse Laplace Transform

We can return from the Laplace transform to the function of time f (t) by the following inverse transformation:

$$f(t) = \mathcal{L}^{-1}[F(p)] = \frac{1}{2\pi j}\int_{-\infty}^{+\infty} e^{-pt}F(p)dp \tag{3.4}$$

where the integration path can be chosen any in the complex plane provided it remains in the convergence domain of F(p).

2. Properties of the Laplace Transform

2.1. Addition

The Laplace transform of a sum of functions $f_1(t)$ And $f_2(t)$ is equal to the sum of their Laplace Transforms.

$$\mathcal{L}(f_1 + f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_2) \tag{3.5}$$

2.2. Multiplication by a constant

$$\mathcal{L}(cf) = c.\mathcal{L}(f) \tag{3.6}$$

2.3. Linearity

The properties of addition and multiplication by a constant when combined lead to the fact that the Laplace transform is a linear transformation:

$$\mathcal{L}\left(\sum_{k=1}^{n} c_k f_k\,(t)\right) = \sum_{k=1}^{n} c_k \mathcal{L}(f_k\,(t)) \tag{3.7}$$

Example :

Determine the Laplace transform of the function f(t) = coswt. This is obtained using the exponential expression.

$$f(t) = coswt = \frac{e^{jwt} + e^{-jwt}}{2}$$

By applying the Laplace transform and the linearity property, we have:

$$\mathcal{L}(f(t)) = F(p) = \mathcal{L}\left(\frac{e^{jwt} + e^{-jwt}}{2}\right) = \frac{1}{2}\mathcal{L}(e^{jwt}) + \frac{1}{2}\mathcal{L}(e^{-jwt}) = \frac{p}{p^2 + w^2}$$

2.4. Derivatives

The first derivative is obtained by : $\mathcal{L}\big(f^{'}(t)\big) = \mathcal{L}\big(pf(t) - f(0)\big) = pF(p) - f(0)$

The second derivative: $\mathcal{L}\big(f^{''}(t)\big) = \mathcal{L}(p^2 f(t) - pf(0) - f^{'}(0)) = p^2F(p) - pf(0) - f^{'}(0)$

The third derivative: $\mathcal{L}\big(f^3(t)\big) = \mathcal{L}(p^3 f(t) - p^2 f(0) - pf^{'}(0) - f^{''}(0)) = p^3F(p) - p^2F(p) - pf'(0) - f^{'}{'}(0)$

Generalization to order derivatives n:

suppose that f(t), and its derivatives $f_k(t)$, For k=1,2,..,notare piecewise continuous and have an exponential order to infinity. So we have:

$$\mathcal{L}(f^{(n)}(t)) = \mathcal{L}(p^n f(t) - p^{n-1}f(0) - p^{n-2}f(0) - \cdots - f^{n-1}(0)) \qquad (3.9)$$

If we consider the initial values all zero, we:

$$\mathcal{L}(f^{(n)}(t)) = p^n\mathcal{L}(f) = p^nF(p)$$

2.5. Initial Value Theorem

We can determine the value of the function f(t) at the origin if we know the limit at infinity of its Laplace transform.

$$f(0^+) = \lim_{s\to\infty} pF(p) \qquad (3.10)$$

2.6. Final Value Theorem

We can determine the value of the function f(t) to infinity if we know the limit for s → 0 its Laplace transform.

$$f(\infty) = \lim_{s\to 0} pF(p) \qquad (3.11)$$

2.7. Delay or delay or rule of translation in t

If $\mathcal{L}(f(t) = F(p))$ so $\mathcal{L}(f(t-T)) = e^{-pT}F(p). e^{-pT}$ is called the delay factor.

2.8. Rule of complex translation in p

$$\mathcal{L}\big(e^{-aT}f(t)\big) = F(p+a)$$

Example :

$$\mathcal{L}(e^{-aT}\cos wt) = \frac{p+a}{(p+a)^2+w^2}$$

2.9. Product of two functions

$$\mathcal{L}\big(f_1(t).f_2(t)\big) = \frac{1}{2\pi j}\int_{c-jw}^{c+jw} F_1(w).F_2(w)dw$$

2.10. Convolution product

$$\mathcal{L}^{-1}\big(F_1(p).F_2(p)\big) = \int_{0^+}^{t} f_1(\tau).f_2(t-\tau)d\tau = \int_{0^+}^{t} f_2(\tau).f_1(t-\tau)d\tau \quad (3.12)$$

2.11. Either f(t) a piecewise continuous function on [0, A] (for all A > 0) and has exponential order at infinity. So, we have: $\mathcal{L}\big(t^n f(t)\big) = (-1)^n F^{(n)}(p)$

Where $F^{(n)}$ is the derivative of order of the function F.

2.12. Either f(t) a piecewise continuous function on [0,A] (for all A >0) and has an order exponential to infinity. Suppose that the limit $\lim_{s\to 0^+}\frac{f(t)}{t}$, is finished. So, we have:

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_{p}^{\infty}\mathcal{L}\big(f(\tau)\big)d\tau \quad (3.13)$$

2.13. Similarity rule (Change of scale)

Either g(t) = f(at) (a > 0), then $\mathcal{L}(f(at)) = \frac{1}{a}F(\frac{p}{a})$

3. Special functions

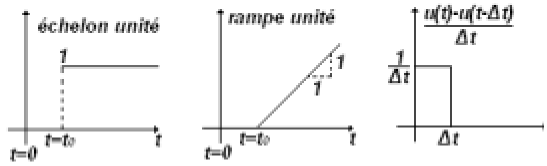
In the study of systems and the differential equations which are used to describe them, we use a particular family of functions, the **singular functions** which are functions of functions **or Distributions**. To fully understand these singular functions, they must be studied within the framework of the theory of distributions, which is a theory that generalizes the theory of functions.

The most frequently used distributions are the unit level distribution (Heaviside distribution). The unit impulse distribution (Dirac distribution) and the unit slope distribution.

3.1. Unit scale function (Heaviside Distribution)

We call **associated unit step function** at t_0, the function of time noted $u(t-t_0)$ and defined by :

$$\mu(t-t_0) = \begin{cases} 1, & si\ t > t_0 \\ 0, & si\ t < t_0 \end{cases}$$



The Laplace transform of the unit level: $\mathcal{L}\big(\mu(t-t_0)\big) = \int_{t_0}^{\infty} e^{-pt} dt = \frac{e^{-pt_0}}{p}\ \forall p > 0$

For the particular case or t_0= 0, we write: $\mathcal{L}\big(\mu(t)\big) = \frac{1}{p}$

3.2. Unit impulse function (Dirac distribution)

We can define the pulse unit $\delta(t)$ as a null function everywhere on $\mathbb{R}$ except for one point t_0 or it takes an infinite value.

$$\delta(t) = \begin{cases} \infty, & t = t_0 \\ 0, & ailleurs \end{cases}$$

The Dirac Distribution can be approximated by the signal represented in Figure 3.1, if we make tend ε towards 0,δ_1 does not tend towards a limit in the sense of functions, but in the sense of distributions because δ_1(t) is not differentiable at the two points of discontinuity. This limit is δ(t) , which is called the Dirac distribution

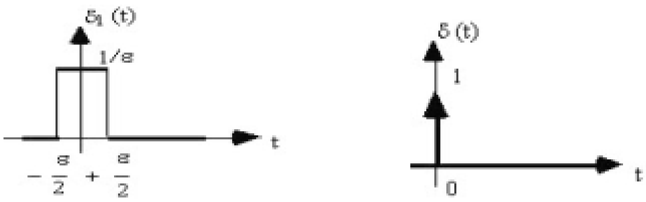


Figure 3.1. The Dirac Distribution.

The Dirac distribution can be obtained as the derivative of the Heaviside distribution. The Laplace transform of the Dirac Distribution is equal to unity: $\mathcal{L}\big(\delta(t)\big) = F(p) = 1$. It is obtained by the following operations:

$$\mathcal{L}\big(\delta(t)\big) = F(p) = \int_0^{\infty} e^{-pt}.\frac{1}{\varepsilon} dt = \frac{1}{\varepsilon}\int_0^{\varepsilon} e^{-pt} dt = \left[\frac{e^{-pt}}{-p}\right]_0^{\varepsilon} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{e^{-p\varepsilon}}{-p} + \frac{1}{p}\right] = \lim_{\varepsilon\to 0}\left[\frac{1-e^{-p\varepsilon}}{p\varepsilon}\right] = 1$$

(We use limited developments or the Hospital rule). We observe that the surface is equal to 1 whatever ε therefore: $\int_{-\infty}^{+\infty} \delta(t) dt = 1$

This function (distribution) also has the following particularity:

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0).$$

3.3. Power function

Lets $t^n U(t) = \begin{cases} t^n, & t \geq 0 \\ 0, & t < 0 \end{cases}$; So let's $F(p) = \int_0^{+\infty} e^{-pt} t^n dt = I_n$ calculate

Let's pose the change of variables: $u = t^n, du = n.t^{n-1}dt$ et $dv = e^{-pt}dt, v = \frac{e^{-pt}}{-p}$; from where : $I_n = \left[t^n \frac{e^{-pt}}{-p}\right] + \frac{n}{p}\int_0^{+\infty} e^{-pt} t^{n-1} dt$ (the first hook is invalid)

from where : $I_n = \frac{n}{p} I_{n-1}$ and so : $I_0 = \frac{1}{p}$; $I_1 = \frac{1}{p^2}$; $I_2 = \frac{2}{p^3}$; ... ; $I_n = \frac{n!}{p^{n+1}}$

from where : $F(p) = \mathcal{L}\left(t^n u(t)\right) = \frac{n!}{p^{n+1}}$ $(n \in \aleph)$

4. Laplace method (Operational calculation)

Using Laplace's method to solve differential equations is called operational calculus or symbolic calculus. It allows knowing the complete solution of a linear system subjected to a wide variety of any transient or periodic signals. The treatment is generally done in four steps as we will see in the following example:

1) - We establish the differential equation to solve.

2) - We apply the derivative and other properties of the Laplace transform to the differential equation considered. By this transformation, we pass from the time domain into the (complex) Laplace domain.

3) - We determine the solution Y(p) in the Laplace plane which we develop in simple terms.

4) - The solution remains to be determined y(p). To do this, we perform the inverse transformation of Y(p) using the transformation table.

4.1. The expansion of y(p) in simple functions

To be able to invert the Laplace transform, which is expressed as: Y(p) =N(p)/D(p), we decompose the equation obtained into a product of factors. Depending on the form of decomposition obtained, we distinguish three cases.

4.1.1. The poles of Y(p) are all simple

Assuming that D(p) has poles $p_0, p_1, p_2, ..., p_n$. we can write Y(p) Under the form :

$$Y(p) = \frac{A}{p-p_0} + \frac{B}{p-p_1} + \cdots + \frac{R}{p-p_n} \tag{3.14}$$

- We know the time response for each term of the sum, we just need to determine the coefficients A_1, A_2, A_n. To do this, we can proceed using the identification method or better still by using residue decomposition techniques. Using the residue technique, we proceed as follows: to determine A we multiply both sides of the equation by p - p_0 then we make it tender p towards p_0. We proceed in the same way for the other coefficients. Here is an example illustration of this technique.

Either p_1, p_2 poles of Y(p)

$$Y(p) = \frac{1}{(p-p_1)(p-p_2)} = \frac{A}{p-p_1} + \frac{B}{p-p_2}$$

The coefficient that we want to determine A, we multiply Y(p) by $(p-p_1)$ as following :

$$Y(p)(p-p_1) = \frac{A(p-p_1)}{p-p_1} + \frac{B(p-p_1)}{p-p_2}$$

We make s tend towards s_1 as following : $A = \lim_{p \to p_1} H(p)(p-p_1) = A.1 + B.0 = \frac{1}{p_1-p_2}$

Likewise for B; we find: $B = \lim_{p \to p_2} H(p)(p-p_2) = A.0 + B.1 = \frac{1}{p_2-p_1}$

Finally knowing that $A/(p-p_0)$ is the transform of $Ae^{p_0 t}$, we obtain the solution :

$$y(t) = \frac{1}{(p_1-p_2)}[e^{p_1 t} - e^{p_2 t}]$$

4.1.2. If there is a multiple pole

If a function with a complex variable has a simple pole, $H(p) = \frac{A}{p-a}$ we obtain A by:

$$A = \lim_{p \to a} H(p)(p-a)$$

If H (p) has a multiple pole of order n: $H(p) = \frac{A}{(p-a)^2}$; we determine A using the expression:

$$A = \lim_{p \to a}[\frac{1}{(n-1)} \frac{d^{n-1}}{dp^{n-1}}((p-a)^n H(p))]$$

Example :

$$H(p) = \frac{1}{(p-p_1)^2(p-p_2)} = \frac{A}{(p-p_1)^2} + \frac{B}{p-p_2}$$

Let

$$A = \lim_{p \to p_1} [\frac{d}{dp}((p-p_1)^2 \frac{1}{(p-p_1)^2(p-p_2)})] = -\frac{1}{(p_1-p_2)^2}$$

$$B = \lim_{z \to z_2} [\frac{1}{(z-z_2)^2} \frac{(z-z_2)}{(z-z_2)}] = \frac{1}{(z-z_2)^2}$$

Generally speaking, if:

$$Y(p) = \frac{N(p)}{(p+p_0)^q D_1(p)}$$

$$Y(p) = \frac{B_0}{(p-p_0)^q} + \frac{B_1}{(p-p_0)^{q-1}} + \cdots + \frac{B_{q-1}}{p-p_0} + \frac{N_1(p)}{D_1(p)}$$

4.1.3. The poles are complex conjugates

$Y(p) = \frac{N(p)}{(p+p_0)(p-p_0)}$ with $p_0 = a + jb \ et \ p_0^* = a - jb$

We will then have $Y(p) = \frac{A_0}{p+p_0} + \frac{A_1}{p-p_0}$ with $A_0 = Ae^{j\varphi} \ and \ A_1 = Ae^{-j\varphi}$

The corresponding coefficients of the decomposition into simple fractions will also be complex conjugates (A and A^x). The solution contains oscillatory terms:

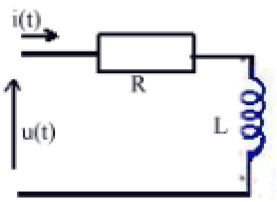
$A.e^{j\phi}.e^{(a+jb)t} + A.e^{-j\phi}.e^{(a-jb)t} = 2A.e^{at}.\cos(bt+\phi)$ and with $\emptyset = \arg(A)$

We are therefore in the presence of a damped oscillation if a is negative, of an amplified (divergent) oscillation if a is positive, of a simple sinusoid if a=0.

4.2. Applications of the Laplace transform

We consider the response of the system corresponding to the circuit above, subjected to a signal step (assumed unitary) U(t). The Laplace transform of the step signal is: $\mathcal{L}[u(t)] = \frac{1}{p}$

For t>0 we have : u(t) =E=constant. We are looking for the current i(t) which circulates in the circuit of the figure below:



1- Ohm's law allows us to write the differential equation:$u(t) = R.i(t) + L.\frac{di}{dt}$

2- We apply the Laplace transformation, in each of these elements taken separately remembering F'(p) =pF(p) ; Which give : $U(p) = R.I(p) + L.p.I(p)$

Replacing U(p) by E/p, the differential equation is expressed in Laplace space by:

$$U(p) = \frac{E}{p} = [R + L.p]I(p)$$

3- We deduce I(p) which we break down into simple terms, namely:

$$I(p) = \frac{E}{p}\frac{1}{R+L.p} = \frac{E/L}{p(p+R/L)} = \frac{A}{p} + \frac{B}{p+R/L}$$

4- We apply the rules for determining the coefficients, we obtain: $I(p) = \frac{E}{R}(\frac{1}{p} - \frac{1}{p+R/L})$ The transform table gives us the solution: $i(t) = \frac{E}{R}(1 - e^{\frac{-R}{L}t})$

5. Modeling

Automation is the science studying automation and dealing with the substitution of automatic mechanisms for all operations capable of being carried out by humans. This science was formerly called cybernetics. Among the components of this science, we will be particularly interested in(automatic) control of continuous dynamic processes.

In this context, we distinguish automatic linear or non-linear, continuous (analog control) or discrete time (digital control).

You should know that the order (orenslavement) of a physical process requires:

- the identification (behavior model) or modeling (knowledge model) of its dynamic behavior → equation;

- the synthesis of a control law → transfer function and Laplace transformation;

- the physical implementation of this control law → correction.

The modeling of a physical system involves a system of differential equations. Its resolution (more or less difficult) allows the determination of transient regimes of the dynamic system. These regimes can also be determined using operational calculation based on the Laplace transformation.

Definition of the Laplace transform

Either *f*(t) a causal function, so the Laplace transform of f is $F(p) = \int_0^{+\infty} e^{-pt} f(t)dt$. We say that F (p) is the image of f (t) in the symbolic domain and f (t) is the image of F(p) in the time domain. We call Laplace transformation the application such as $\mathcal{L}(f) = F$.

Properties

We suppose that *F*(p) And *G*(p) are the images of *f*(t) And *g*(t), two causal functions. **Uniqueness :** Any time function *f*(t) has a unique image *F*(p); and reciprocally.

Linearity

- The image of 0 is 0.
- The image of k.f(t) is k.F(p).
- The image off(t) +g(t) is F(p) +G(p).

Derivation – Integration

- The image of f'(t),the derivative of f is pF(p) -f(0) with most often, f(0) = 0.
- The image of $\int_0^t f(u)\, du$,is the primitive of f is $\frac{1}{p} F(p)$.

Scale factor : The image of $f(a.t) is\ \frac{1}{a} F(\frac{p}{a})$

Delay and Amortization

- The image of $f(t-\tau) is\ e^{-\tau p} F(p)$.

- The image of $e^{\omega t} f(t) is\ F\ (p+\omega)$.

Final and initial value theorem

- $f(0) = \lim_{t\to 0} f(t) = \lim_{t\to\infty} pF(p)$
- $\lim_{t\to+\infty} f(t) = \lim_{p\to 0^+} pF(p)$

Convolution:

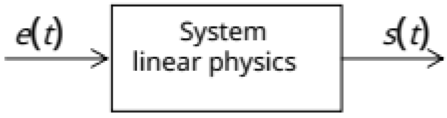
The image of the convolution product $f(t) * g(t) is\ F(p) \times G(p)$

6. Usual transformations

Symbolic image	Temporal image of causal functions
$\frac{1}{p}$	Échelon
1	Dirac
$\frac{1}{p^2}$	Rampe
$\frac{1}{p+a}$	e^{-at}
$\frac{w}{p^2+w^2}$	$\sin(wt)$
$\frac{w}{p^2-w^2}$	$\sinh(wt)$
$\frac{w}{(p+a)^2+w^2}$	$e^{-at}\sin(wt)$
$\frac{p}{p^2+w^2}$	$\cos(wt)$
$\frac{p}{p^2-w^2}$	$\cosh(wt)$
$\frac{p+a}{(p+a)^2+w^2}$	$e^{-at}\cos(wt)$
$\frac{n!}{p^{n+1}}$	t^n
$\frac{1}{p(1+\tau p)}$	$1-e^{-\frac{t}{\tau}}$
$\frac{1}{(p+a)^2}$	te^{-at}

7. Transfer function

A transfer is the transmittance $H(p) = \frac{s(p)}{E(p)}$ of a linear system generating a signal of exit s(t) from an entry e(t).

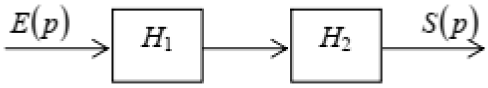


Knowing that $s(t) = \mathcal{L}^{-1}\{H(p)E(p)\}$, we have $s(t) = f(t) * e(t)$ where h(t) is the answer impulse of the physical system.

8. Transactions on transfers

8.1. Cascading transfers

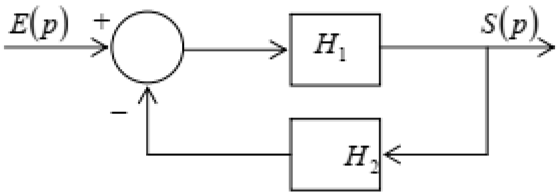
Consider the following diagram:



Transmittance is calculated : $H(p) = H_1(p) \times H_2(p)$.

8.2. Reactive transfers

Consider the following diagram:



The transmittance is written:

$$H(p) = \frac{H_1(p)}{1+H_1(p)H_2(p)}.$$

<u>Representation of the frequency response of a transfer</u>

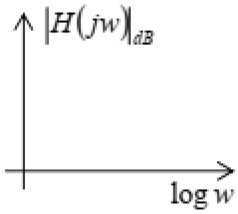
The frequency response translates the behavior in sinusoidal regime, it is obtained by replacing *p* by *jw* Or *w* is the pulsation expressed in *rad/s*. The frequency *f* And the period *T* are linked to the pulsation by the relationships :$\omega = 2\pi f = \frac{2\pi}{T}$

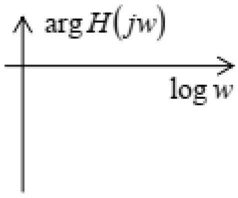
A frequency response can be characterized by its module and by his argument:

$H(j\omega) = p(\omega)e^{j\theta(\omega)}$. The gain can be expressed in decimal or in decibel $p_{dB} = 20logp$

9. Bode diagram

Bode's diagram consists of a gain diagram in dB and a phase diagram



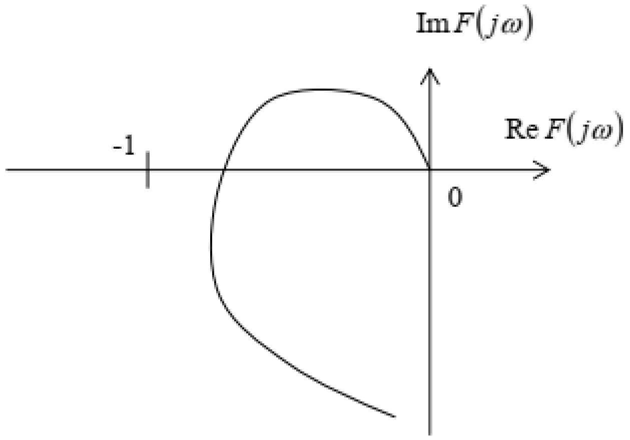


In rd Or d°. To trace them, we use the asymptotic Bode diagrams. They are defined piecewise by studying locally (for given frequency bands) the asymptotic behavior of the frequency response.

We reduce ourselves, as much as possible, to a product of transfers from 1erand 2thorder, for which we know the asymptotic diagrams well; and we proceed by superposition to obtain the final Bode diagram.

10. Nyquist Plan – Hall Abacus

Frequency representation of $F(j\omega)$ on *Nyquist's* place:



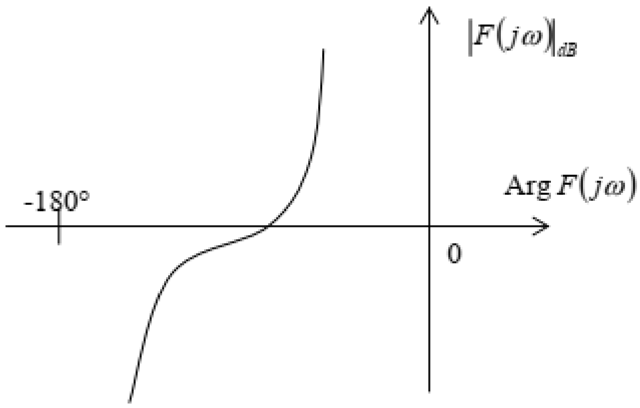
suppose that $F(j\omega)$ designates the FTBO of a process. We then have: $FTBF(jw) = \frac{A(jw)}{1+F(jw)}$.
By therefore, the point (-1, 0) is critical at the Nyquist location.

Furthermore, the distance to the critical point on the plane gives :$|1 + F(j\omega)|$

Hall's abacus gives the modules and arguments of $\frac{A(j\omega)}{1+F(j\omega)}$ $for\ a$ $F(j\omega)$ given. We can subsequently deduce the modules and arguments of the FTBF calculating $B(j\omega)$, and applying the following relationship:$FTBF\ (j\omega) = \frac{F(j\omega)}{1+F(j\omega)} \times B^{-1}(j\omega)$.

11. Black Plane (Nichols) – Black Abacus

Frequency representation of $F(j\omega)$ on Black's place:



The critical point is now (-180°, 0 dB).

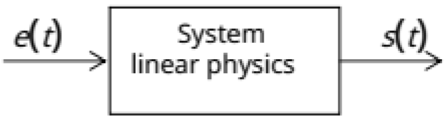
For a $F(j\omega)$ given, **Black's abacus** gives the modules and arguments of $\frac{F(j\omega)}{1+F(j\omega)}$

12.Temporal and frequency analysis

We notice *F* (p) there Laplace transform of a function *f* (t)

12.1. Transform of a differential equation with constant coefficients, transfer function

We consider the following physical system



in which e(t) And s(t) are linked by a linear differential equation with constant coefficients without constant term:

$$a_n \frac{d^n s(t)}{dt^n} + \cdots + a_1 \frac{ds(t)}{dt} + a_0 s(t) = b_n \frac{d^m e(t)}{dt^m} + \cdots + b_1 \frac{de(t)}{dt} + b_0 e(t)$$

The system is then linear.

If the initial conditions following are zero :

$$n \frac{d^{n-1} s(0)}{dt^n} = \cdots = \frac{ds(0)}{dt} = s(0) = \frac{d^{m-1} e(0)}{dt^m} = \cdots = \frac{de(0)}{dt} = e(0) = 0$$

The transform of the differential equation without constant term is expressed by:

$$a_n \frac{d^n s(t)}{dt^n} + \cdots + a_1 \frac{ds(t)}{dt} + a_0 s(t) = b_n \frac{d^m e(t)}{dt^m} + \cdots + b_1 \frac{de(t)}{dt} + b_0 e(t)$$

⇓Laplace and zero initial conditions

$$a_n p^n S(p) + a_{n-1} p^{n-1} S(p) + \cdots + a_0 S(p) = b_m p^m E(p) + b_{m-1} p^{m-1} E(p) + \cdots + b_0 E(p)$$

We can then define a transfer function :

$$\frac{S(p)}{E(p)} = \frac{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_0}{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_0}$$

that's to say :

$$S(p) = \frac{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_0}{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_0} E(p) = F(p).E(p)$$

n = system order.

$K_p = \lim_{p \to 0} F(p)$: system static gain.

$F(p)$ can be put in the form $F(p) = \frac{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_0}{(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_0).p^{\alpha}}$, α =system class .

(possibly α= 0) = number of integrations in the system)

12.2. First order systems

These are systems such as: $\tau.s^{'}(t) = K.e(t)$

$$H(p) = \frac{S(p)}{E(p)} = \frac{K}{1 + \tau p}$$

The transfer function is written

K: Static gain

τ : time constant

Etude temporelle

<u>Réponse à un échelon</u> $e(t)=E_0.u(t)$;

$$E(p)=\frac{E_0}{p}$$

$$S(p)=\frac{K}{1+\tau.p}.\frac{E_0}{p}$$

Avec les théorèmes sur les limites, il vient :

$$s(0+)=\lim_{p\to\infty} p.S(p)=\lim_{p\to\infty}\frac{E_0.k}{1+\tau.p}=0$$

$$s(\infty)=\lim_{p\to 0} p.S(p)=\lim_{p\to 0}\frac{E_0.K}{1+\tau.p}=E_0.K$$

$$\dot{s}(0+)=\lim_{p\to\infty} p.(p.S(p)-s(0+))=\lim_{p\to\infty} p\frac{E_0.K}{1+\tau.p}=\frac{E_0.K}{\tau}$$

<u>Expression de *s(t)*</u>

$$s(t)=E_0.K.(1-e^{-t/\tau}u(t)$$

<u>Temps de réponse</u> t_r à 5% tel que $s(t_r)=0{,}95.s(\infty)$

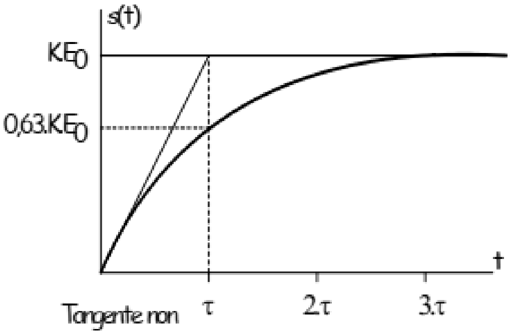
$$s(t_r)=E_0.K(1-e^{-\frac{t_r}{\tau}})=0{,}95.s(\infty)=0{,}95.E_0.K$$

$$\Rightarrow 1-e^{-\frac{t_r}{\tau}}=0{,}95$$ ou

$t_{r=-\tau.\ln(0{,}05)\approx 2{,}99573.\tau}$

D'où : $t_r=3\tau$

<u>Tracé de la réponse indicielle</u>



Tangente non nulle à l'origine caractéristique des premiers ordres

Etude fréquentielle

$$H(j.\omega)=\frac{K}{1+\tau.j.\omega}$$

<u>Gain</u>

$$G(\omega)=|H(j.\omega)|=\frac{K}{\sqrt{1+(\tau.\omega)^2}}$$

<u>Gain en décibels</u>

$$G_{dB}(\omega)=20.\text{Log}|H(j.\omega)|=20.\text{Log}K-20.\text{Log}\sqrt{1+(\tau.\omega)^2}$$

$$G_{dB}(\omega\approx 0)=\approx 20.\text{Log}K$$

$$G_{dB}(\omega=1/\tau)=\approx 20.\text{Log}K-20.\text{Log}\sqrt{2}=Gdb(0)-3dB$$

$$G_{dB}(\omega\to\infty)=20.\text{Log}K-20.\text{Log}\tau.\omega=20.\text{Log}\frac{K}{\tau}-20.\text{Lc}$$

Déphasage

$$\phi(\omega)=\text{Arg}H(j.\omega)=\text{Arg}\frac{K}{1+\tau.j.\omega}=\text{Arctan}(\tau.\omega)$$

$$\phi(\omega\approx 0)\approx 0$$

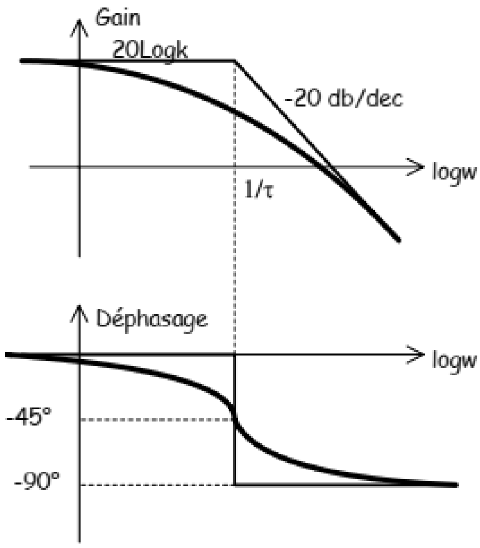
$$\phi(\omega=1/\tau)=-45°$$

$$\phi(\omega\to\infty)\approx -90°$$

<u>Pulsation de coupure</u> telle que $G(\omega_c)=1$ si K>1 :

$$\frac{K}{\sqrt{1+(\tau.\omega_c)^2}}=1 \text{ ou } K^2=1+(\tau.\omega_c)^2 \Rightarrow \omega_c=\sqrt{\frac{K^2-1}{\tau}}$$

Tracé du diagramme de Bode



CHAPTER IV

The z-transform

Introduction

The z-transform of a sequence x[n] is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

The z-transform can also be thought of as an operator Z{·} that transforms a sequence to a function:

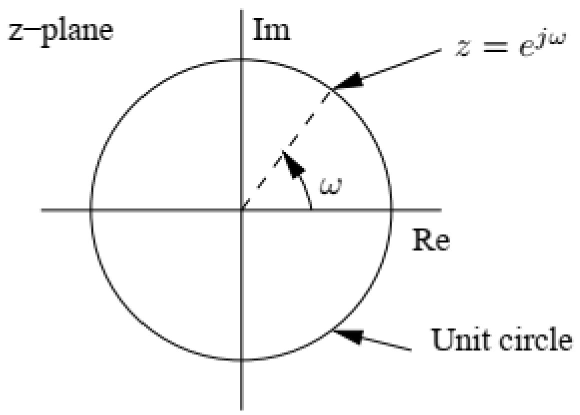
$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z).$$

In both cases z is a continuous complex variable.

We may obtain the Fourier transform from the z-transform by making the substitution $z = e^{j\omega}$. This corresponds to restricting |z| = 1. Also, with $z = re^{j\omega}$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n}.$$

That is, the z-transform is the Fourier transform of the sequence $x[n]r^{-n}$. For r = 1 this becomes the Fourier transform of x[n]. The Fourier transform therefore corresponds to the z-transform evaluated on the unit circle:



The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation.

The Fourier transform does not converge for all sequences — the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. The set of values of z for which the
z-transform converges is called the **region of convergence (ROC)**.

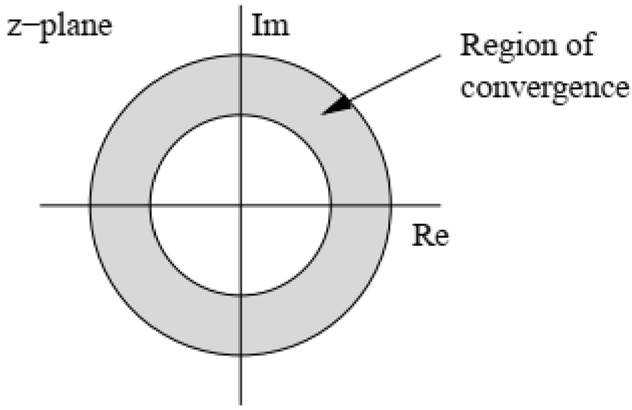
The Fourier transform of x[n] exists if the sum $\sum_{n=-\infty}^{\infty} |x[n]|$ converges. However, the z-transform of x[n] is just the Fourier transform of the sequence
$x[n]r^{-n}$. The z-transform therefore exists (or converges) if :

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$$

This leads to the condition

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

for the existence of the z-transform. The ROC therefore consists of a ring in the z-plane:



In specific cases the inner radius of this ring may include the origin, and the outer radius may extend to infinity. If the ROC includes the unit circle |z| = 1, then the Fourier transform will converge.

Most useful z-transforms can be expressed in the form : $X(Z) = \frac{P(Z)}{Q(Z)}$

where P (z) and Q(z) are polynomials in z. The values of z for which P (z) = 0 are called the **zeros** of X(z), and the values with Q(z) = 0 are called the **poles**. The zeros and poles completely specify X(z) to within a multiplicative constant.

Example: right-sided exponential sequence

Consider the signal $x[n] = a^n u[n]$. This has the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

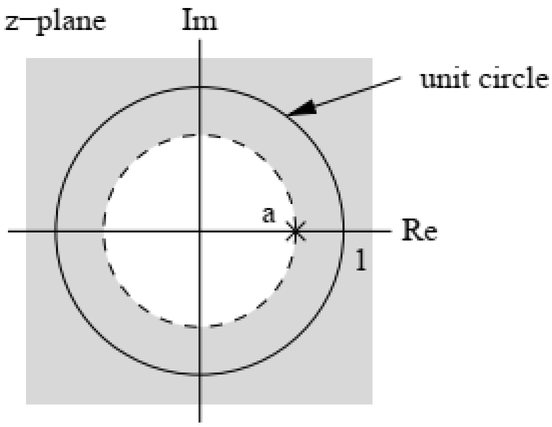
Convergence requires that :

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty,$$

which is only the case if $|az^{-1}| < 1$, or equivalently $|z| > |a|$. In the ROC, the series converges to:

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|,$$

since it is just a geometric series. The z-transform has a region of convergence for any finite value of a.



The Fourier transform of x[n] only exists if the ROC includes the unit circle, which requires that $|a| < 1$. On the other hand, if $|a| > 1$ then the ROC does not include the unit circle, and the Fourier transform does not exist. This is consistent with the fact that for these values of a the sequence $a^n u[n]$ is exponentially growing, and the sum therefore does not converge.

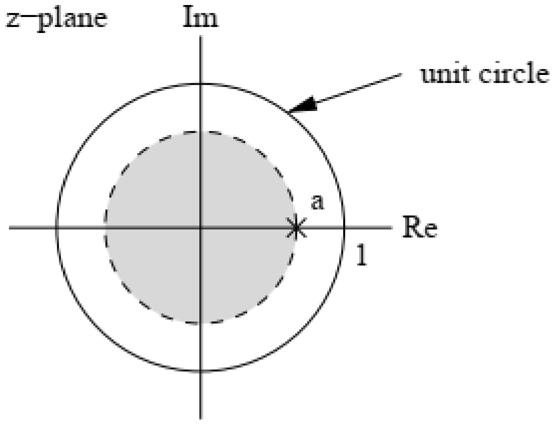
Example: left-sided exponential sequence

Now consider the sequence $x[n] = -a^n u[-n - 1]$. This sequence is left-sided because it is nonzero only for $n \leq -1$. The z-transform is :

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$
$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n.$$

For $|a^{-1}z| < 1$, or $|z| < |a|$, the series converges to :

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|.$$



Note that the expression for the z-transform (and the pole zero plot) is exactly the same as for the right-handed exponential sequence — *only the region of convergence is different*. Specifying the ROC is therefore critical when dealing with the z-transform.

Example: sum of two exponentials

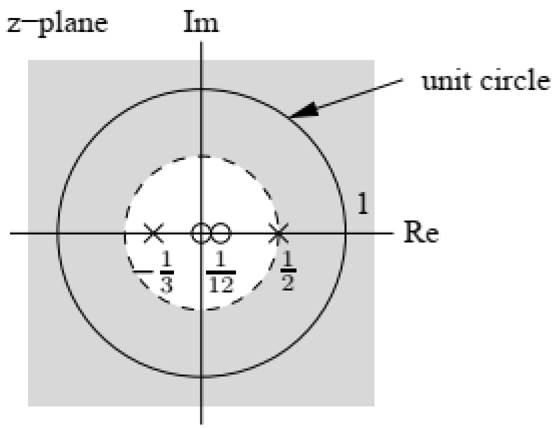
The signal $x[n] = (1/2)^n u[n] + (-1/3)^n u[n]$ is the sum of two real exponentials. The z-transform is :

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n . \end{aligned}$$

From the example for the right-handed exponential sequence, the first term in this sum converges for $|z| > 1/2$, and the second for $|z| > 1/3$. The combined transform X(z) therefore converges in the intersection of these regions, namely when $|z| > 1/2$. In this case :

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}.$$

The pole-zero plot and region of convergence of the signal is :



Example: finite length sequence

The signal :

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

has z-transform :

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$
$$= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}.$$

Since there are only a finite number of nonzero terms the sum always converges when az^{-1} is finite. There are no restrictions on a ($|a| < \infty$), and the ROC is the entire z-plane with the exception of the origin z = 0 (where the terms in the sum are infinite). The N roots of the numerator polynomial are at :

$$z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N-1.$$

since these values satisfy the equation $z^N = a^N$. The zero at k = 0 cancels the pole at z = a, so there are no poles except at the origin, and the zeros are at :

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1.$$

1. Properties of the region of convergence

The properties of the ROC depend on the nature of the signal. Assuming that the signal has a finite amplitude and that the z-transform is a rational function:

• The ROC is a ring or disk in the z-plane, centered on the origin ($0 \le r_R < |z| < r_L \le \infty$).

• The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform includes the unit circle.

• The ROC cannot contain any poles.

• If x[n] is finite duration (ie. zero except on finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$), then the ROC is the entire z-plane except perhaps at z = 0 or z = ∞.

• If x[n] is a right-sided sequence then the ROC extends outward from the outermost finite pole to infinity.

• If x[n] is left-sided then the ROC extends inward from the innermost nonzero pole to z = 0.

• A two-sided sequence (neither left nor right-sided) has a ROC consisting of a ring in the z-plane, bounded on the interior and exterior by a pole (and not containing any poles).

• The ROC is a connected region.

4. The inverse z-transform

Formally, the inverse z-transform can be performed by evaluating a Cauchy integral. However, for discrete LTI systems simpler methods are often sufficient.

4.1 Inspection method

If one is familiar with (or has a table of) common z-transform pairs, the inverse can be found by inspection. For example, one can invert the z-transform :

$$X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2},$$

using the z-transform pair :

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad \text{for } |z| > |a|.$$

By inspection we recognise that :

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Also, if X(z) is a sum of terms then one may be able to do a term-by-term inversion by inspection, yielding x[n] as a sum of terms.

2.2 Partial fraction expansion

For any rational function we can obtain a partial fraction expansion, and identify the z-transform of each term. Assume that X(z) is expressed as a ratio of polynomials in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

It is always possible to factor X(z) as :

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

where the ck's and dk's are the nonzero zeros and poles of X(z).

• If M < N and the poles are all first order, then X(z) can be expressed as :

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

In this case the coefficients A_k are given by :

$$A_k = (1 - d_k z^{-1})X(z)\big|_{z=d_k}.$$

• If M ≥ N and the poles are all first order, then an expansion of the form :

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

can be used, and the B_r's be obtained by long division of the numerator by the denominator. The A_k's can be obtained using the same equation as for M < N.

• The most general form for the partial fraction expansion, which can also deal with multiple-order poles, is :

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}.$$

Ways of finding the Cm's can be found in most standard DSP texts. The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, and invert to terms of the form $B_r\delta[n - r]$. The fractional terms :

$$\frac{A_k}{1 - d_k z^{-1}}$$

correspond to exponential sequences. For these terms the ROC properties must be used to decide whether the sequences are left-sided or right-sided.

Example: inverse by partial fractions Consider the sequence x[n] with z-transform

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}, \qquad |z| > 1.$$

Since M = N = 2 this can be expressed as :

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

The value B_0 can be found by long division:

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \overline{) \; z^{-2} + 2z^{-1} + 1} \\ \underline{z^{-2} - 3z^{-1} + 2} \\ 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

The coefficients A_1 and A_2 can be found using :

$$A_k = (1 - d_k z^{-1})X(z)\big|_{z=d_k},$$

$$A_1 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1}}\right|_{z^{-1}=2} = \frac{1 + 4 + 4}{1 - 2} = -9$$

$$A_2 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1}=1} = \frac{1 + 2 + 1}{1/2} = 8.$$

Therefore :

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

Using the fact that the ROC is $|z| > 1$, the terms can be inverted one at a time by inspection to give :

$$x[n] = 2\delta[n] - 9(1/2)^n u[n] + 8u[n].$$

2.3 Power series expansion

If the z-transform is given as a power series in the form :

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \ldots + x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots, \end{aligned}$$

then any value in the sequence can be found by identifying the coefficient of the appropriate power of z^{-1}.

Example: finite-length sequence

The z-transform :

$$X(z) = z^2(1 – 1/2z^{-1})(1 + z-1)(1 - z^{-1})$$

can be multiplied out to give :

$$X(z) = z^2 – 1/2z - 1 +1/ 2z^{-1}.$$

By inspection, the corresponding sequence is therefore :

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently :

$$x[n] = 1\delta[n + 2] - 1 /2\delta[n + 1] - 1\delta[n] + 1/ 2\delta[n - 1].$$

Example: power series expansion

Consider the z-transform :

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Using the power series expansion for log(1 + x), with |x| < 1, gives :

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

The corresponding sequence is therefore

$$x[n] = \begin{cases} (-1)^{n+1} \frac{a^n}{n} & n \geq 1 \\ 0 & n \leq 0. \end{cases}$$

Example: power series expansion by long division

Consider the transform :

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

Since the ROC is the exterior of a circle, the sequence is right-sided. We therefore divide to get a power series in powers of z^{-1}:

$$\begin{array}{r|l} & 1+az^{-1}+a^2z^{-2}+\cdots \\ \hline 1-az^{-1}) & 1 \\ & \underline{1-az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1}-a^2z^{-2}} \\ & \qquad\qquad a^2z^{-2}+\cdots \end{array}$$

Or :

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots .$$

Therefore $x[n] = a^n u[n]$.

Example: power series expansion for left-sided sequence

Consider instead the z-transform :

$$X(z) = \frac{1}{1-az^{-1}}, \qquad |z| < |a|.$$

Because of the ROC, the sequence is now a left-sided one. Thus we divide to obtain a series in powers of z:

$$\begin{array}{r|l} & -a^{-1}z-a^{-2}z^2-\cdots \\ \hline -a+z) & z \\ & \underline{z-a^{-1}z^2} \\ & \quad az^{-1} \end{array}$$

Thus $x[n] = -a^n u[-n-1]$.

3. Properties of the z-transform

In this section, if X(z) denotes the z-transform of a sequence x[n] and the ROC of X(z) is indicated by R_x, then this relationship is indicated as :

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad \text{ROC} = R_x.$$

Furthermore, with regard to nomenclature, we have two sequences such that :

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z), \qquad \text{ROC} = R_{x_1}$$
$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z), \qquad \text{ROC} = R_{x_2}.$$

3.1. Linearity

The linearity property is as follows:

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \quad \text{ROC contains} R_{x1} \cap R_{x1}.$$

3.2. Time shifting

The time-shifting property is as follows:

$$x[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z), \quad \text{ROC} = R_x.$$

(The ROC may change by the possible addition or deletion of $z = 0$ or $z = \infty$.) This is easily shown:

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)}$$
$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-n_0} X(z).$$

Example: shifted exponential sequence

Consider the z-transform :

$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \frac{1}{4}.$$

From the ROC, this is a right-sided sequence. Rewriting,

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4} z^{-1}} = z^{-1} \left(\frac{1}{1 - \frac{1}{4} z^{-1}} \right), \qquad |z| > \frac{1}{4}.$$

The term in brackets corresponds to an exponential sequence $(1/4)^n u[n]$. The factor z^{-1} shifts this sequence one sample to the right. The inverse z-transform is therefore :

$$x[n] = (1/4)^{n-1} u[n - 1].$$

Note that this result could also have been easily obtained using a partial fraction expansion.

3.3 Multiplication by an exponential sequence

The exponential multiplication property is :

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \qquad \text{ROC} = |z_0| R_x,$$

where the notation $|z_0|R_x$ indicates that the ROC is scaled by $|z_0|$ (that is, inner and outer radii of the ROC scale by $|z_0|$). All pole-zero locations are similarly scaled by a factor z_0: if X(z) had a pole at z = z1, then $X(z/z_0)$ will have a pole at $z = z_0 z_1$.

• If z_0 is positive and real, this operation can be interpreted as a shrinking or expanding of the z-plane — poles and zeros change along radial lines in the z-plane.

• If z_0 is complex with unit magnitude ($z_0 = e^{j\omega 0}$) then the scaling operation corresponds to a rotation in the z-plane by and angle ω_0. That is, the poles and zeros rotate along circles centered on the origin. This can be interpreted as a shift in the frequency domain, associated with modulation in the time domain by $e^{j\omega 0n}$. If the Fourier transform exists, this becomes :

$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

Example: exponential multiplication

The z-transform pair :

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

Can be used to determine the z-transform of $x[n] = r_n \cos(\omega_0 n)u[n]$. Since :

$\cos(\omega_0 n) = 1/2e^{j\omega 0n} + 1/2e^{-j\omega 0n}$, the signal can be rewritten as :

$$x[n] = 1/2(re^{j\omega 0})^n u[n] + 1/\,2(re^{-j\omega 0})^n u[n].$$

From the exponential multiplication property,

$$\frac{1}{2}(re^{j\omega_0})^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1/2}{1-re^{j\omega_0}z^{-1}}, \qquad |z| > r$$

$$\frac{1}{2}(re^{-j\omega_0})^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1/2}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r,$$

$$X(z) = \frac{1/2}{1-re^{j\omega_0}z^{-1}} + \frac{1/2}{1-re^{-j\omega_0}z^{-1}}, \qquad |z| > r$$

$$= \frac{1-r\cos\omega_0 z^{-1}}{1-2r\cos\omega_0 z^{-1}+r^2 z^{-2}}, \qquad |z| > r.$$

3.4 Differentiation

The differentiation property states that :

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

This can be seen as follows: since :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n},$$

we have :

$$-z\frac{dX(z)}{dz} = -z \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$

Example: second order pole

The z-transform of the sequence :

$$x[n] = na^n u[n]$$

can be found using :

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad |z| > a,$$

To be :

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > a.$$

3.5. Conjugation

This property is :

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x.$$

3.6. Time reversal

Here :

$$x^*[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}.$$

The notation 1/Rx means that the ROC is inverted, so if Rx is the set of values such that $r_R < |z| < r_L$, then the ROC is the set of values of z such that $1/r_l < |z| < 1/r_R$.

Example: time-reversed exponential sequence

The signal $x[n] = a^{-n}u[-n]$ is a time-reversed version of $a^n u[n]$. The z-transform is therefore :

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

3.7. Convolution

This property states that :

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains} R_{x_1} \cap R_{x_2}.$$

Example: evaluating a convolution using the z-transform

The z-transforms of the signals $x_1[n] = a_n u[n]$ and $x_2[n] = u[n]$ are :

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

For $|a| < 1$, the z-transform of the convolution $y[n] = x_1[n] * x_2[n]$ is :

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z - a)(z - 1)}, \qquad |z| > 1.$$

Using a partial fraction expansion,

$$Y(z) = \frac{1}{1 - a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right), \qquad |z| > 1,$$

so

$$y[n] = \frac{1}{1 - a}(u[n] - a^{n+1}u[n]).$$

3.8. Initial value theorem

If $x[n]$ is zero for $n < 0$, then :

$$x[0] = \lim_{z \to \infty} X(z).$$

Some common z-transform pairs are:

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

3.9. Relationship with the Laplace transform

Continuous-time systems and signals are usually described by the Laplace transform. Letting z = esT , where s is the complex Laplace variable

$$s = d + j\omega,$$

we have :

$$z = e^{(d+j\omega)T} = e^{dT} e^{j\omega T}.$$

Therefore :

$$|z| = e^{dT} \text{ and } \measuredangle z = \omega T = 2\pi f / fs = 2\pi\omega/\omega s,$$

where ω_s is the sampling frequency. As ω varies from ∞ to ∞, the s-plane is mapped to the z-plane:

- The jω axis in the s-plane is mapped to the unit circle in the z-plane.
- The left-hand s-plane is mapped to the inside of the unit circle.
- The right-hand s-plane maps to the outside of the unit circle.

CHAPITRE V

Convolution product and signal correlation

Introduction

Signal processing is a growing discipline, it consists of a set of theories and methods, relatively independent of the signal processed, allowing the creation, analysis, modification, classification and finally recognition of signals. Its applications are numerous in fields as varied as telecommunications, sound processing, speech processing, radar, sonar, biomedical, imaging, etc. Generally speaking in the fields of electronics and IT.

In this chapter, we first present the main definitions and generalities concerning convolution and correlation as two signal processing techniques which are particularly described. Their developments and implementation represent the bulk of this chapter.

1. Convolution Product

1.1. Convolution product formulation

The convolution product has very important properties. The convolution product is an operation which associates two functions *h* And *e* of the same variable, a functions of the same variable on the same infinite domain. Functions can take complex values and we note *s=h*e*. By choosing *t* as a common variable, the operation is defined by:

$$s(t) = \int_{-\infty}^{+\infty} h(t-\tau)e(\tau)d\tau \tag{5.1}$$

In mathematics, the convolution product is a bilinear operator and a commutative product, denoted by "*", which has two functions *f* And g on the same infinite domain, corresponds to another function $f * g$ on this domain, which at every point of it is equal to the integral over the entire domain (or the sum if it is discrete) of one of the two functions around this point, weighted by the other function around the origin; the two functions being traversed in opposite directions to each other (necessary to guarantee commutativity).

The convolution product generalizes the idea of a rolling average and is the representation mathematics of the notion of linear filter. It applies both to temporal data (in signal processing for example) and to spatial data (in image processing). In statistics, we use a very similar formula to define cross-correlation.

1.1.1. Definition of convolution product

Context : Consider a Linear and Time Invariant system (SLIT) defined by its response to a Dirac impulse, $h(t)$.

Objective : Determine the output signal when applying a signal x(t) entrance.

Solution : It is possible to demonstrate that the operation performed by the system is a convolution product.

The convolution product of two real or complex functions *f* and *g*, is another function, which is generally noted $f * g$, and which is defined by:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

τ : is the dummy variable of the convolution product.

We can consider this formula as a generalization of the idea of a moving average.

For this definition to have meaning, it is necessary that *f* And *g* satisfy certain assumptions; for example, if these two functions are integrable in the Lebesgue sense (i.e. the integral of their module is finite), their convolution product is defined for almost all tand is itself integrable.

Example :

Either x(t) =y(t) = Π_1 is two width door functions L=1. The convolution product expressed in the form:

$$(x * y)(t) = \int_{-1/2}^{1/2} \Pi_1(t-\tau)d\tau = \int_{t-1/2}^{t+1/2} \Pi_1(u)du$$

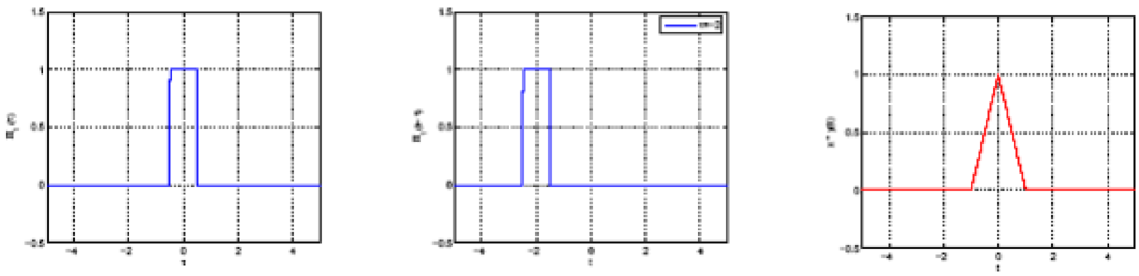
$si\ t < -1\ out > 1, \quad (x * y)(t) = 0$

$si\ -1 < t \leq 0\ , \quad \int_{-1/2}^{t+1/2} \Pi_1(u)du = 1 + t$

$si\ 0 \leq t < 1\ , \quad \int_{t-1/2}^{1/2} \Pi_1(u)du = 1 - t$

Finally using the previous equations, we find:

$$(x * y)(t) = \begin{cases} 1 - |t|, & -1 < t < 1 \\ 0, & ailleurs \end{cases}$$



$x(t)$ $y(t)$ $(x * y)(t)$

1.2. Properties of the convolution product:

Commutativity : $f * g = g * f$

Distributivity : $(f * (h + g))(t) = (f * h)(t) + (f * g)(t)$

Associativity : $((f * h) * g))(t) = ((f * (h * g))(t)$

Provided with a neutral element equal to δ: $f * \delta = f$

Integration of a convolution product: $\int_{-\infty}^{+\infty} (f * g)(t)dt = (\int_{-\infty}^{+\infty} f(t)dt) . (\int_{-\infty}^{+\infty} g(t)dt$

Derivation : $(f * g)' = f' * g = f * g'$

Parity of the convolution of two even functions: $(f * g)(t) = (f * g)(-t)$

Convolution product and Fourier transform:

$$F(f * g) = F(f).F(g); f * g = F^{-1}(F(f).F(g))$$

1.3. Convolution product and Dirac momentum

The fact that the Dirac function is the neutral element of the convolution product induces an interesting property concerning the product of a continuous function f(t) by a Dirac comb $\boldsymbol{\delta}_{Te}(t)$. The convolution product in this case is defined by:

$$s(t) = \int_{-\infty}^{+\infty} f(t - \tau)\delta_{Te}(\tau)d\tau = T_e \sum_{n=-\infty}^{+\infty} f(t - nT_e) \quad (5.2)$$

Thus, the convolution product of f(t) by δ_{Te} (t) gives a periodic function which is obtained by summing, over all possible values of n, the function f(t) shifted by n_{Te}, and multiplying the result by Te. This operation is sometimes said to “periodize” f(t).

1.4. Deconvolution

Deconvolution is an algorithmic process intended to reverse the effects of convolution. The concept of deconvolution is widely used in signal processing and image processing, particularly in microscopy and astronomy.

The problem is to determine the solution *f* of an equation of the form: $f * g = h$

We denote here by h a signal as it is acquired and *f* the signal that we wish to estimate or restore, but which was convolved by an impulse response g during acquisition. The impulse response is often (especially in image processing) also called Point Spread Function (PSF).
When dealing with a physical acquisition process, the measurement his often marred by measurement noise ε : $(f * g) + \varepsilon = h$

The deconvolution operation will be made more difficult by the presence of “noise”. Applying the analytical inverse of deconvolution (by convolving with Green's function) will give a poor quality result. It is then necessary to include statistical knowledge of the noise and the signal to improve the result, for example using “Wiener filtering”.

There are therefore a large number of deconvolution methods in signal processing based on different types of priors and therefore adapted to various applications.

2. Correlation Function

In signal processing, it is often necessary to compare two signals, this can be done in several ways. A possible method, which is widely used, is to shift one of the signals (stationary and ergodic) relative to the other, and to measure their similarity as a function of the shift. This is the correlation function (FAC).

We distinguish between auto-correlation (FAC) and inter-correlation (FIC):

- FAC consists of comparing a functionS(t) with itself, during an interval of time, one of which is shifted by a certain value τ.

- FIC sometimes replaced by mutual correlation or cross correlation (in English: Cross correlation) consists of comparing two different functions S(t) And Y(t) one of which is shifted of a certain valueτ.

We can list the different steps and operations involved in the calculation of a correlation function as follows:

- One of the signals is shifted by a certain quantity τ.
- The product of the two signals is carried out sample by sample for all values of the correlation function.
- The values thus obtained are added to obtain a value of the correlation function.

2.1. Autocorrelation function

EitherS(t) a real, stationary and ergodic random process; we distinguish :

2.1.1. Statistical autocorrelation function

The statistical FAC is defined as the mathematical expectation of the product of S(t) And S(t+ τ):

$$\Gamma_{SS}(\tau) = E[S(t)S(t+\tau)] \tag{5.3}$$

2.1.2. Temporal autocorrelation function

The temporal FAC of a finite power signal is given by the time average value of the product of S(t) by S(t+ τ) :

$$\Gamma_{SS}(\tau) = \overline{S(t)S(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} S(t)S(t+\tau)dt \tag{5.4}$$

Generally speaking, FAC is defined as follows:

$$\Gamma_{SS}(\tau) = E[S(t)S(t+\tau)] = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} S(t)S(t+\tau)dt \tag{5.5}$$

This formula reflects the fact that in the case of a stationary process, the statistical FAC is equal to the temporal FAC (this is the case of ergodicity).

2.1.3. Properties of the autocorrelation function

1. The FAC of real signals is also real.

2. According to second-order stationarity, the FAC of real signals is even and depends only on τ :

$$E[S(t)S(t-\tau)] = E[S(t)S(t+\tau)] \qquad (5.6)$$

Where :

$$\Gamma_{SS}(\tau) = \Gamma_{SS}(-\tau) \qquad (5.7)$$

We notice that the autocorrelation function is also symmetric.

3. When the FAC is complex, the real part of the FAC is an even function, while the imaginary part is an odd function

4. The FAC is defined by the dot product as follows:

$$\Gamma_{SS}(\tau) = \prec S, S_\tau \succ = \int_{-\infty}^{+\infty} S(t)S(t+\tau)dt \qquad (5.8)$$

5. The FAC of a stationary process reaches its maximum at $\tau = 0$, with a value always real and non-negative, this value is the upper limit in modulus of the FAC.

According to SCHWARZ, we obtain the following inequality :

$$\Gamma_{SS}(\tau)^2 \leq \Gamma_{SS}(0) \qquad \forall t \qquad (5.9)$$

This value is equal to the signal energy :

$$\Gamma_{SS}(0) = \prec S, S_\tau \succ = \left\| S(t) \right\|^2 = \int_{-\infty}^{+\infty} \left| S(t) \right|^2 \, dt \qquad (5.10)$$

6. The FAC of periodic or continuous signals, is also a periodic function of the same period or continuous.

7. In case the function S(t) is composed by the sum of two functions U(t) And V(t),there FAC of S(t) is defined by the following relation:

$$\Gamma_{SS}(\tau) = \Gamma_{UU}(\tau) + \Gamma_{VV}(\tau) + \Gamma_{UV}(\tau) + \Gamma_{VU}(\tau) \qquad (5.11)$$

8. If S(t) is the product of two functions U(t) And V(t) real and independent, the FAC is equal to the product of the FACs of the two functions :

$$\Gamma_{SS}(\tau) = \Gamma_{UU}(\tau).\Gamma_{VV}(\tau) \qquad (5.12)$$

9. The FAC of a random signal tends to zero when the offset τ increases indefinitely in absolute value.

CHAPITER VI

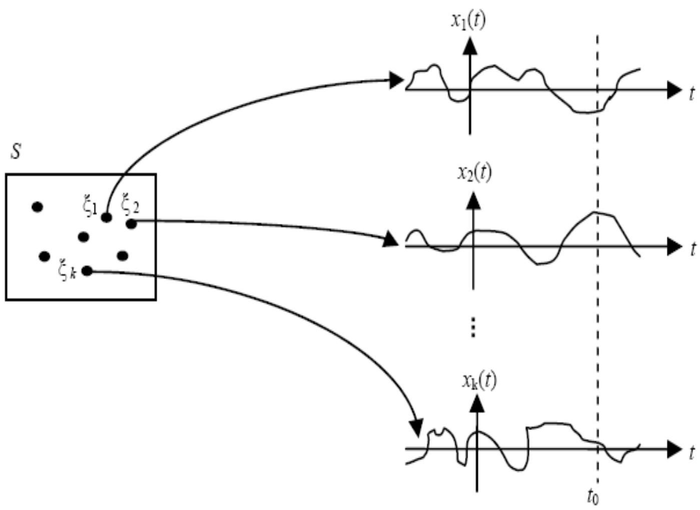
Random Processes

Introduction

In signal theory, we most often study quantities that depend on time and whose evolution seems unpredictable. The study of the sound emitted by vehicles passing on a road gives us a concrete and familiar image of such magnitude (whereas the roll of a die is a static motion independent of time). To model them, we use the notion of the random process which associates with each test an achievement which is no longer a value as in the case of random variables, but a function of time.

We define a random process (PA) as an application which, at each test ω, matches a function of time t. We use the notation $X(t, \xi)$ or more usually $X(t)$in which we omit the statistical dependence on the test ξ.

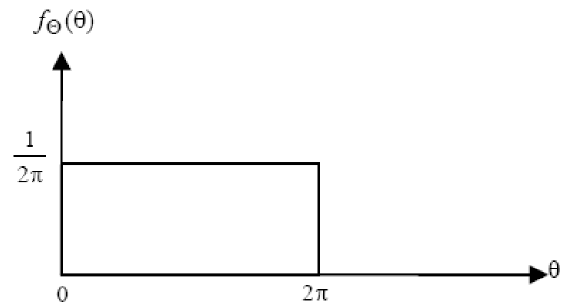
Example of random process



A process can then be seen:

Or, for a fixed test ξ_0, $X(t, \xi_0)$as a function of time which we call trajectory

Or, for a fixed instant t_0, $X(t_0, \xi)$like a random variable and $x(t_0)$becomes a particular value of the va$X$



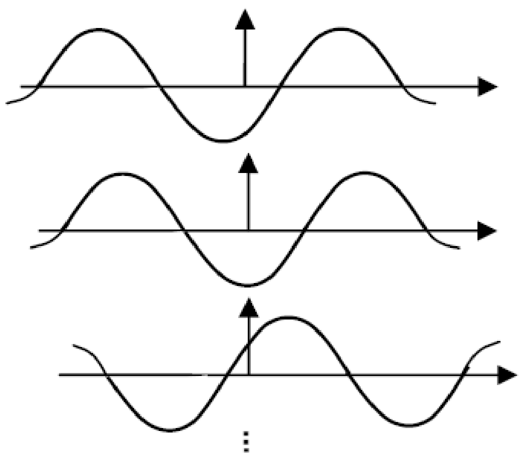
Example 2.1

Consider the random process $X(t) = Acos(\omega t + \Theta)$, where $\Theta$is a uniformly distributed va between 0 and 2π, as is illustrated in Figure 2.1. SO

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & otherwise \end{cases}$$

Such a random process, whose future values are predictive from previous values, is called predictive. Indeed, by setting the value of Θ, to $\pi/4$for example, the function $X(t, \xi_k)$becomes a deterministic function of time, then

$$x_k(t) = Acos\left(\omega t + (\frac{\pi}{4})\right)$$



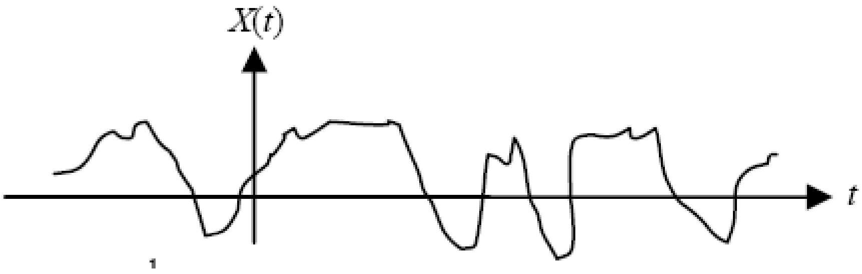
Continuous-time (TC) and discrete-time (TD) random processes

A random process is continuous time (TC), if the time$t \in \boldsymbol{R}$ (t $réel$)

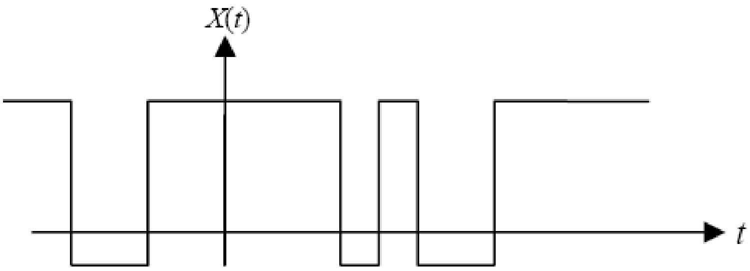
A random process is discrete time (TD), if the time$t \in \boldsymbol{Z}$ (t entire)

In general, we are interested in four types of random processes, depending on the characteristics of time $t$and flow $X(t) = X\ au\ temps\ t$. We will therefore have

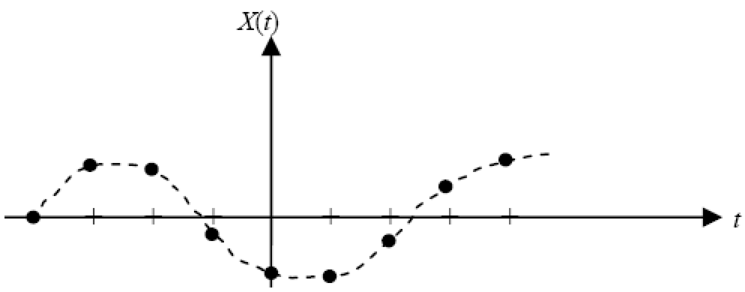
1 ***Continuous state (test) and continuous time*** : in this case, $X(t)$and $t$both are continuous. $X(t)$is called a continuous random process as shown in Figure 3.2



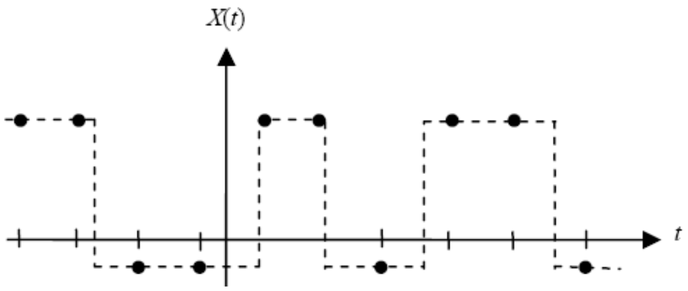
2 Discrete state and continuous time. $X(t)$assumes a set of discrete values, while time $t$is continuous. Such a process is considered a discrete random process. Figure 3.3 illustrates this process



3 Continuous state and discrete time. $X(t)$takes continuous values, while time is a set of discrete values, as shown in Figure 3.4. Such a process is called a continuous random sequence.



4 Discrete state and discrete time. $X(t)\ et\ t$ are both considered a set of discrete values. Such a process is considered a discrete random sequence. Figure 3.5 illustrates this process.



By fixing the time t, the random process $X(t)$becomes a va. In this case, the techniques we use with random variables are valid. Consequently, we can characterize the random process by the distribution of order 1 (the distribution function: cumulative density function):

$$F_X(x;t) \ = P[X(t_0) \leq x]$$

1st order density function :

$$f_X(x;t \ \ ) = \frac{d}{dx} F_X(x;t \ \ )$$

For all values of$t$

2nd order distribution function is the joint distribution of two random variables $X(t_1)$and $X(t_2)$for each $t_1$and t_1. SO :

$$F_X(x_1, x_2; t_1, t_2) = P[X(t_1) \leq x_1 \ \ et \ \ X(t_2) \leq x_2 \ ]$$

2nd order density function is:

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

Normally, the complete probabilistic description of any random process requires knowledge of the distributions from the 1st to the nth order, given by:

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P[X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n \ ]$$

nth order density function is:

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{\partial^n F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)}{\partial x_1 \partial x_2 \ \ldots \ \partial x_n}$$

Expectations

In many situations, only 1st and 2nd order statistics may be necessary to characterize a random process. Given a real random process $X(t)$:

Average$\boldsymbol{m_x(t)}$

$$m_x(t) = E[X(t)] = \int_{-\infty}^{+\infty} x f_X(x;t)dx$$

It is the statistical average, the mathematical expectation or even moment of order 1 of $X(t)$. This quantity is deterministic and depends on$t$

Autocorrelation function $\boldsymbol{R_{xx}(t_1, t_2)}$: also called autocovariance if the process is stationary and centered,

$$R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X_1X_2}(x_1, x_2; t_1, t_2)dx_1 dx_2$$

Stationarity in the wide sense of the processes at TC and TD

Rating : if the process is at **TC** , $t, \tau, t_1, t_2 \in \mathbb{R}$, if the rpoprocess is at **TD** , $t, \tau, t_1, t_2 \in \mathbb{Z}$.

When the autocorrelation function $R_{xx}(t_1, t_2)$only depends on the difference $|t_1 - t_2|$, and the mean $m_x(t)$is constant, we say that the random process is stationary in the broad sense (SSL), that is to say the process verifies the following two properties:

1. The average of the random process is independent of time (constant),
$$E[X(t)] = m_x(t) = m_x$$
2. The aucorrelation function only depends on τ. In this case, $R_{xx}(t_1, t_2)$is written based on a single argument $\tau = t_1 - t_2$. If $t_2 = t$and $t_1 = t + \tau$,
$$R_{xx}(t + \tau, t) = R_{xx}(\tau)$$

Stationarity in the strict sense or strictly stationary (strictly stationary or stationnary in the strict sense)

A random process is strictly stationary if its statistics are invariant with respect to time or a time shift (a time shift in the time origin). A strictly stationary process is also stationary in the broad sense (SSL). The opposite is not true.

Example 2.2

Check whether the random process of Example 2.1 is stationary in the broad sense.

Solution

For a random process to be stationary in the broad sense, it must satisfy two conditions:

1. $E[X(t)] = constante.$
2. $R_{xx}(t+\tau) = R_{xx}(\tau).$

To calculate the average of $X(t)$, we use the concept of a function of a va

$$E[g(\Theta)] = \int_{-\infty}^{+\infty} g(\theta)\, f_\Theta(\theta) d\theta$$

Such that, in this case $g(\theta) = A \cdot cos(\omega t + \theta)$and $f_\Theta(\theta) = \frac{1}{2\pi}$in the interval from 0to$2\pi$

$$E[X(t)] = \int_0^{2\pi} Acos(\omega t + \theta) \frac{1}{2\pi} d\theta = 0$$

The autocorrelation function is:

$$E[X(t+\tau,t)X(t)] = E\{Acos[\omega(t+\tau)+\theta]Acos(\omega t + \theta)\}$$

$$= \frac{A^2}{2} E[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\theta)]$$

Where we used the trigonometric relationship:

$$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$$

The second term is evaluated at zero. So the autocorrelation function is:

$$R_{xx}(t+\tau,t) = \frac{A^2}{2} cos\omega\tau = R_{xx}(\tau)$$

Since the mean is constant and the autocorrelation function depends only on τ, $X(t)$is a wide-sense stationary process.

In the case of two va $X(t)$and $Y(t)$, we say that they are jointly wide-sense stationary in the broad sense if each process is stationary in the broad sense, and

$$R_{xy}(t+\tau,t) = E[X(t+\tau)Y(t)] = R_{xy}(\tau)$$

As $R_{xy}(t_1,t_2)$represented by the cross-correlation function. We also define the **autocovariance function** (the autocovariance function) $C_{xx}(t_1,t_2)$and the intercorrelation function (the cross-covariance function) $C_{xy}(t_1,t_2)$ between $X(t)$and $Y(t)$as follows:

$$C_{xx}(t_1,t_2) = E\{[X(t_1) - m_x(t_1)][X(t_2) - m_x(t_2)]\}$$

$$= E[X(t_1)X(t_2)] - m_x(t_1)m_x(t_2)$$

And

$$C_{xy}(t_1,t_2) = E\{[X(t_1) - m_x(t_1)][Y(t_2) - m_y(t_2)]\}$$

Special cases

- If the process is stationary in the broad sense (SSL) , the autocovariance will be

$$C_{xx}(t_1,t_2) = C_{xx}(\tau) = E[X^0(t+\tau)X^0(t)] = E[X(t+\tau)X(t)] - m_x^2$$

with

$$X^0(t+\tau) = [X(t+\tau) - m_x(t+\tau)]$$

- If, moreover, the process is centered $m_x = 0$, the autocovariance function merges with the autocorrelation function

$$C_{xx}(\tau) = E[X(t+\tau)X(t)] = R_{xx}(\tau)$$

We also have :

$$C_{xx}(\tau) = E[X^0(t)X^0(t-\tau)]$$

And

$$C_{xx}(-\tau) = E[X^0(t)X^0(t+\tau)] = C_{xx}(\tau),$$ $\boldsymbol{C_{xx}(\tau)}$**is even**

Noticed

The autocovariance of a strictly stationary random process $X(t)$depends only on the difference $t_1 - t_2$. The equation (of the autocovariance) also shows that if we know the mean and the autocorrelation function of a random process, we can only determine its autocovariance function. The mean and the autocorrelation function are then sufficient to describe the first two moments of the process.

However, two important points should be noted:

1. The mean and the autocorrelation function provide only a partial description of the process distribution$X(t)$
2. The conditions on the mean and on the autocorrelation function are not sufficient to guarantee whether the process $X(t)$is strictly stationary.

We limit ourselves to stationary 2nd order processes in the broad sense (SSL) (the most common case).

The class of random processes which satisfy the two stationarity conditions are rather called "second order stationaries" or even "stationary in the broad sense". a stationary process is not necessarily strictly stationary, because the two conditions do not imply time invariance of the joint distribution (of dimension k)

Complex Random Process

If $Z(t)$is a complex random process such that $Z(t) = X(t) + jY(t)$

The autocorrelation and autocovariance functions are

$$R_{zz}(t_1, t_2) = E[Z(t_1)Z^*(t_2)]$$

And

$$C_{zz}(t_1, t_2) = E[\{Z(t_1) - m_z(t_1)\}\{Z(t_2) - m_z(t_2)\}^*]$$

Or$*$ denotes the complex conjugate and $m_z(t)$the mean of $Z(t)$.

The cross-correlation and the cross-covariance functions between the complex random process $Z(t)$and another complex random process $W(t)$, $W(t) = U(t) + jV(t)$, are

$$R_{zw}(t_1, t_2) = E[Z(t_1)W^*(t_2)]$$

And

$$C_{zw}(t_1, t_2) = E[\{Z(t_1) - m_z(t_1)\}\{W(t_2) - m_w(t_2)\}^*]$$

Example 2.3

Let $I(t)$and $Q(t)$two random processes such that

$I(t) = X\cos\omega t + Y\sin\omega tAndQ(t) = Y\cos\omega t - X\sin\omega t$

where $X$and $Y$are two uncorrelated random variables with zero mean. The values of the square means of $X$and $Y$are $E[X^2] = E[Y^2] = \sigma^2$. Give the intercorrelation functionR_{iq}

Solution

$$R_{iq}(t+\tau, t) = E[I(t+\tau)Q(t)]$$

$$= E[\{X\cos(\omega t + \omega\tau) + Y\sin(\omega t + \omega\tau\}\{Y\cos\omega t - X\sin\omega t\}]$$

$$= E[XY]\cos(\omega t + \omega\tau)\cos\omega t - E[X^2]cos(\omega t + \omega\tau)\sin\omega t + E[Y^2]sin(\omega t + \omega\tau)\cos\omega t$$

$$-E[XY]\sin(\omega t + \omega\tau)\sin\omega t$$

Using trigonometric relations and as long as $X$and $Y$are uncorrelated and have zero means ($E[XY] = E[X] = E[Y] = 0$), we find:

$$R_{iq}(t+\tau, t) = -\sigma^2\sin\omega\tau$$

Example 2.3'

Consider $N(t)$a random process defined as follows:

$$N(t) = U \cdot exp[-|t|] + V$$

Or U et $V$are two independent random variables with zero mean$E[U] = E[V] = 0$

1. Determine $E[N(t)]$; calculate $E[N(0)]$ and$E[N(2)]$
2. Calculate $E[N^2(0)]$ and$E[N^2(2)]$
3. Determine the autocorrelation function of this process
4. Is the process stationary
5. Calculate the correlation coefficient between $N(0)$and$N(2)$

Solution

$$E[N(t)] = E[U]e^{-|t|} + E[V]$$

If $E[U] = E[V] = 0$then$E[N(t)] = 0$

$$E[N(0)] = E[U] + E[V]$$

$$E[N^2(0)] = E[(U+V)^2] = E[U^2] + E[V^2] + E[U]E[V]$$

If more $\sigma_U^2 = \sigma_V^2 = 1$then:$E[N^2(0)] = 2$

$$E[N^2(2)] = E[(Ue^{-2} + V)^2] = E[U^2]e^{-4} + E[V^2] + E[U]E[V]e^{-2}$$

If more $\sigma_U^2 = \sigma_V^2 = 1$then$E[N^2(2)] = 1 + e^{-4}$

The autocorrelation function

$$R_{NN}(t_1, t_2) = E[N(t_1) \cdot N(t_2)] = e^{-|t_1|}e^{-|t_1|}E[U^2] + E[U]E[V]\left(e^{-|t_1|} + e^{-|t_2|}\right) + E[V^2]$$

If more $\sigma_U^2 = \sigma_V^2 = 1$then:$R_{NN}(t_1, t_2) = e^{-|t_1|}e^{-|t_1|} + 1$

Since the autocorrelation function $R_{NN}(t_1, t_2)$depends on time, the process is not stationary.

The correlation coefficient:

$$\rho_N(0,2) = \frac{E[(N(0) - E(N(0)) \cdot (N(2) - E(N(2))]}{\sigma_N(0)\sigma_N(2)} = \frac{E[(U+V)(Ue^{-2} + V)]}{\sigma_N(0)\sigma_N(2)}$$

$$\rho_N(0,2) = \frac{E[U^2]e^{-2} + E[U]E[V](1 + e^{-2}) + E[V^2]}{\sigma_N(0)\sigma_N(2)}$$

If more$\sigma_U^2 = \sigma_V^2 = 1$ $E[U] = E[V] = 0$SO

$$\rho_N(0,2) = \frac{1 + e^{-2}}{\sqrt{2(1 + e^{-4})}}$$

Properties of the Autocorrelation Function (AFC)

By notation convention, we recall the autocorrelation function of a stationary random process $X(t)$:

$$R_{xx}(\tau) = E[X(t+\tau)X(t)] \quad for \;\; all \;\; t$$

This FAC has several properties:

The average of the quadratic value of the process can be obtained from $R_{xx}(\tau)$, by setting $\tau = 0$,

$$R_{xx}(0) = E[X^2(t)]$$

The autocorrelation function is an even function:

$$R_{xx}(\tau) = R_{xx}(-\tau)$$

We can also define the autocorrelation function $R_{xx}(\tau)$, as follows:

$$R_{xx}(\tau) = E[X(t)X(t-\tau)]$$

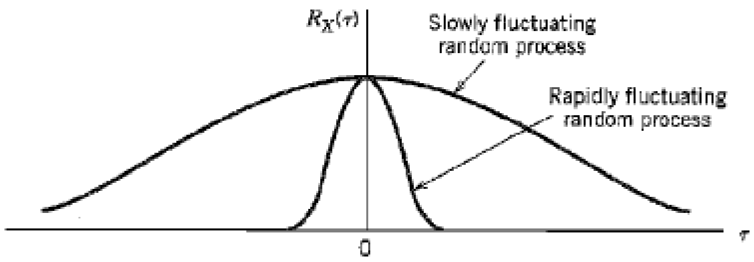


Fig. illustration of autocorrelation functions of slowly and rapidly fluctuating random processes

The autocorrelation function is max at the origin ($\tau = 0$), that is to say:

$$|R_{xx}(\tau)| \leq R_{xx}(0)$$

To demonstrate this property, consider the following nonnegative quantity:

$$E\left[(X(t+\tau) \pm X(t))^2\right] \geq 0$$

$$\Rightarrow E\left[X^2(t+\tau)\right] \pm 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0$$

$$\Rightarrow 2R_{xx}(0) \pm 2R_{xx}(\tau) \geq 0$$

We can write

$$-R_{xx}(0) \leq R_{xx}(\tau) \leq R_{xx}(0)$$

That's to say

$$|R_{xx}(\tau)| \leq R_{xx}(0)$$

$$R_{xx}(t_2, t_1) = R_{xx}^*(t_1, t_2)$$

If $X(t)$is real, then the autocorrelation function is symmetric in the plane(t_1, t_2)

$$R_{xx}(t_2, t_1) = R_{xx}(t_1, t_2)$$

The value of the root mean square of the random process $X(t)$is always positive, then:

$$R_{xx}(t_1, t_1) = E[X(t_1)X^*(t_1)] = E[|X(t)|^2] \geq 0$$

If $X(t)$is real, the value of the root mean square $E[X^2(t)]$is always non-negative.

$$|R_{xx}(t_1, t_2)| \leq \sqrt{R_{xx}(t_1, t_1)R_{xx}(t_2, t_2)}$$

This is Schwartz's inequality. It can be written as follows:

$$|R_{xx}(t_1, t_2)|^2 \leq E[|X(t_1)|^2]E[|X(t_2)|^2]$$

$$\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} a_i a_j^* R_{xx}(t_i, t_j) \geq 0$$

For any sequence of values $a_0, a_2, \ldots, a_{N-1}$and any sequence of instants $t_0, t_2, \ldots, t_{n-1}$. Then, the autocorrelation is a nonnegative definite function. We can write the positivity property as follows:

$A^T\boldsymbol{R}A \geq 0$($A^H\boldsymbol{R}A \geq 0$in the complex case), the superscripts T and H indicate transposition and transposition-conjugation respectively.

Example for N=3:

$$A^T\boldsymbol{R}A = [a_0 \;\; a_1 \;\; a_2]\begin{bmatrix} R_{xx}(0) & R_{xx}(1) & R_{xx}(2) \\ R_{xx}(1) & R_{xx}(0) & R_{xx}(1) \\ R_{xx}(2) & R_{xx}(1) & R_{xx}(0) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \geq 0$$

In the complex case,

$$A^H\boldsymbol{R}A = \left[a_0^* \quad a_1^* \;\; a_2^*\right]\begin{bmatrix} R_{xx}(0) & R_{xx}(1) & R_{xx}(2) \\ R_{xx}(-1) & R_{xx}(0) & R_{xx}(1) \\ R_{xx}(-2) & R_{xx}(-1) & R_{xx}(0) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \geq 0$$

Due to the staionarity of the random process, the matrix $\boldsymbol{R}$is such that the parallels to the main diagonal consist of equal terms. This form of matrix is called the ***Toeplitz matrix.***

Properties of the Intercorrelation Function (FAC)

Consider the two random processes $X(t)$and Y(t).

$$R_{xy}(t_1, t_2) = R^*_{yx}(t_2, t_1)$$

And on the other hand

$$R_{xy}(t_1, t_2) = E\ [X(t_1)Y(t_2)]$$

$$R_{yx}(t_1, t_2) = E\ [Y(t_1)X(t_2)]$$

In matrix form, we write:

$$\boldsymbol{R}(t_1, t_2) = \begin{bmatrix} R_{xx}(t_1, t_2) & R_{xy}(t_1, t_2) \\ R_{yx}(t_1, t_2) & R_{yy}(t_1, t_2) \end{bmatrix}$$

This matrix is called the correlation matrix.

If the two processes $X(t)$and $Y(t)$are each stationary, and jointly stationary, then the correlation matrix can be written as follows:

$$\boldsymbol{R}(\tau) = \begin{bmatrix} R_{xx}(\tau) & R_{xy}(\tau) \\ R_{yx}(\tau) & R_{yy}(\tau) \end{bmatrix}$$

Or$\tau = t_1 - t_2$

If $X(t)$and Y(t) are real processes:

$$R_{xy}(t_1, t_2) = R_{yx}(t_2, t_1)$$

In general, $R_{xy}(t_1, t_2)$and $R_{yx}(t_2, t_1)$are different

$$\left|R_{xy}(t_1, t_2)\right| = E[X(t_1)]E[Y(t_2)]$$

$$\leq \sqrt{R_{xx}(t_1, t_1)R_{yy}(t_2, t_2)} = \sqrt{E[X^2(t_1)]E[Y^2(t_2)]}$$

X(t)and Y(t) are wide sense staionary (SSL),

The autocorrelation function is an even function of τ :

$$R_{xx}(\tau) = R_{xx}(-\tau)$$

$$R_{xx}(0) = E[|X|^2(t)]$$

As long as $X(t)$it is real:

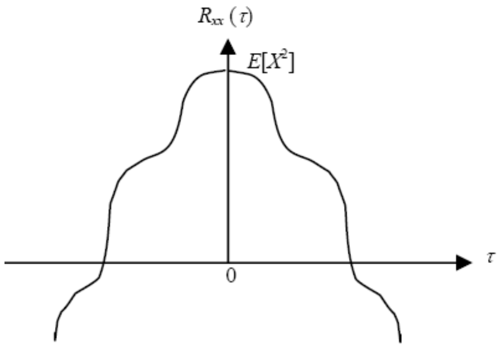
$$R_{xx}(0) = E[X^2(t)] = E[X^2(t)] - m_x^2 + m_x^2 = \sigma_x^2 + m_x^2 \geq 0$$

Remember that we have the max from $R_{xx}(\tau)$to $\tau = 0$. The further we move away from the origin, the $R_{xx}(\tau)$faster it decreases. Yes $\tau \longrightarrow \infty$, the two observations can be uncorrelated. In this case, the autocovariance function tends to 0.

$$\lim_{\tau \longrightarrow \infty} C_{xx}(\tau) = E[\{X(t+\tau) - m_x\}\{X(t) - m_x\}] = R_{xx}(\tau) - m_x^2 = 0$$

Or

$$\lim_{\tau \longrightarrow \infty} R_{xx}(\tau) = |m_x|^2$$



Example of an autocorrelation function

We obtain the same properties if $X(t), Y(t)$they are jointly stationary in the wide sense,

$$R_{xy}^*(\tau) = R_{xy}(-\tau)$$

$$\left|R_{xy}(\tau)\right|^2 \leq R_{xx}(0)R_{yy}(0)$$

$$R_{xy}(0) = R_{xy}^*(0)$$

$$\left|R_{xy}(\tau)\right| \leq \frac{R_{xx}(0) + R_{xy}(0)}{2}$$

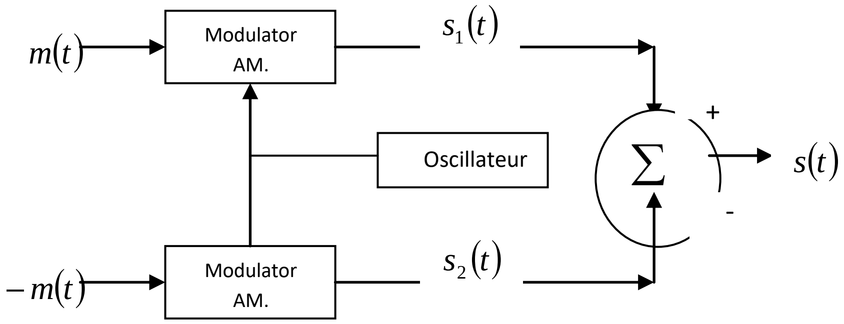
Example 2.4

Consider a pair of quadrature modulated processes $X_1(t), X_2(t)$that are related to the stationary process $X(t)$:

$$X_1(t) = X(t)cos(2\pi f_c t + \Theta)$$

$$X_2(t) = X(t)sin(2\pi f_c t + \Theta)$$

The real example of this process is the generation of double sideband in AM modulation (DSB-SC), or we recall the diagram for greater clarity in the presentation:



where $f_c$is the carrier frequency, and $\Theta$is a random variable (va) uniformly distributed over the interval $[0, 2\pi]$. However, $\Theta$is independent of $X(t)$. the intercorrelation of $X_1(t), X_2(t)$is

$$R_{x_1x_2}(\tau) = R_{12}(\tau) = E[X_1(t)X_2(t-\tau)]$$

$$= E[X(t)X(t-\tau)cos(2\pi f_c t + \Theta)sin(2\pi f_c t - 2\pi f_c \tau + \Theta)]$$

$$= \frac{1}{2}E[X(t)X(t-\tau)sin(4\pi f_c t - 2\pi f_c \tau + 2\Theta) - X(t)X(t-\tau)sin(2\pi f_c \tau)]$$

$$= \frac{1}{2}E[X(t)X(t-\tau)](E[sin(4\pi f_c t - 2\pi f_c \tau + 2\Theta)] - E[sin(2\pi f_c \tau)])$$

$$= -\frac{1}{2}R_{xx}(\tau)E[sin(2\pi f_c \tau)]$$

$$\left(E[sin(4\pi f_c t - 2\pi f_c \tau + 2\Theta)] = \frac{1}{2\pi}\int_0^{2\pi} sin(4\pi f_c t - 2\pi f_c \tau + 2\Theta)d\theta = 0\right)$$

Note that at $\tau = 0$,$R_{12}(\tau) = -\frac{1}{2}R_{xx}(\tau)E[sin(0)] = E[X_1(t)X_2(t)] = 0$

This shows the values obtained by the simultaneous observations $X_1(t), X_2(t)$are orthogonal.

Some random processes

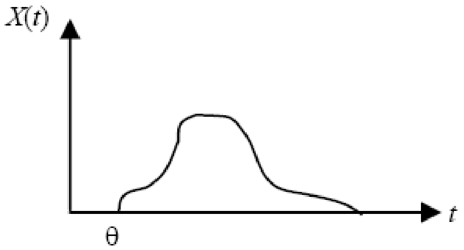
In this paragraph, we will study certain random processes which can characterize some applications.

A Single Pulse of Known Shape but Random Amplitude and Arrival Time

In Radar and Sonar applications, the received signal (a return signal) can be characterized as a random process "one pulse", but with random amplitude and arrival time. The impulse can be expressed by:

$$X(t) = A\,S(t-\Theta)$$

where $A$and $\Theta$are statistically independent random variables, and $s(t)$is a deterministic function. An example of this function is shown in Figure



The average value of particular random process is given by:

$$E[X(t)] = E[A\,S(t-\Theta)]$$

As long as $A$and $\Theta$are statistically independent, we will have:

$$E[X(t)] = E[A]E[\,S(t-\Theta)] = E[A]\int_{-\infty}^{+\infty} s(t-\theta)f_{\Theta}(\theta)d\theta$$

The integral $\int_{-\infty}^{+\infty} s(t-\theta)f_{\Theta}(\theta)d\theta$is simply the convolution of the momentum $s(t)$with the density function of Θ. SO :

$$E[X(t)] = E[A]s(t) * f_{\Theta}(\theta)$$

By analogy,

The autocorrelation function is given by:

$$R_{xx}(t_1,t_2) = E[A^2] \int_{-\infty}^{+\infty} s(t_1-\theta)\, s(t_2-\theta) f_\Theta(\theta) d\theta$$

If the arrival time is known with a certain value θ_0, then the average and autocorrelation functions become $X(t)$:

$$E[X(t)] = E[A]\, s(t-\theta_0)]$$

And

$$R_{xx}(t_1,t_2) = E[A^2] s(t_1-\theta_0) s(t_2-\theta_0)$$

Particular case:

The arrival time can be uniformly distributed over the interval $[0,T]$. The mean and autocorrelation functions:

$$E[X(t)] = E[A]E[S(t-\Theta)] = \frac{E[A]}{T}\int_0^T s(t-\theta)d\theta$$

And

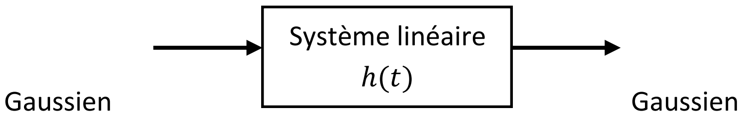
$$R_{xx}(t_1,t_2) = \frac{E[A]}{T}\int_0^T s(t_1-\theta)\, s(t_2-\theta)d\theta$$

Gaussian Process

The random process $X(t)$is Gaussian if the random variables $X(t_1)$, $X(t_2), \ldots$, $X(t_n)$are jointly Gaussian for all possible values of $n$and $t_1, t_2, \ldots, t_n$. As long as the multiple Gaussian random variables depend only on the mean vector and the covariance matrix of $n$random values, we observe that $X(t)$stationary in the broad sense (SSL). If $X(t)$is a Gaussian random process applied to an invariant linear system with an impulse response $h(t)$, as shown in Figure 2.4, then the random process:

$$Y(t) \;=\; \int_{-\infty}^{+\infty\infty} x(t-\tau)h(\tau)d\tau$$

is also Gaussian.



Example 2.5

Let $X(t)$a random SSL process, Gaussian with zero mean, be an input to a quadratic detector (a square law detector), a nonlinear system without memory.

1. Check that the output is no longer Gaussian.
2. Determine the autocorrelation function $R_{yy}(\tau)$, the output and the variance.

Solution

The density function of the input is:

$$f_X(x;t) = f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

Using the fundamental theorem:

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \Lambda + \frac{f_X(x_n)}{|g'(x_n)|} + \Lambda$$

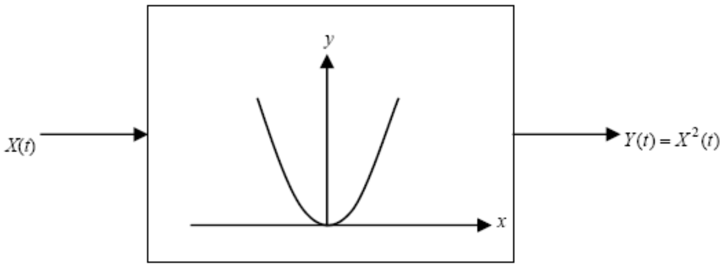
$Y = X^2 \Rightarrow X = \pm\sqrt{Y}$, we therefore have two roots:$x_1 = +\sqrt{y},\ x_2 = -\sqrt{y}$

$$g'(x) = 2x \Rightarrow g'(x_1) = 2(\sqrt{y}), \qquad g'(x_2) = -2(\sqrt{y})$$

SO :

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

$$f_Y(y;t) = f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2\sigma^2}, & y \geq 0 \\ 0, & otherwise \end{cases}$$



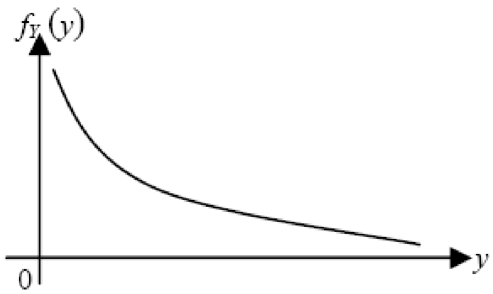
Quadratic detector

We observe that the output of the nonlinear system is no longer Gaussian.

The autocorrelation function of the output $Y(t) = X^2(t)$is given by:

$$R_{yy}(t+\tau, t) = E[Y(t+\tau)Y(t)] = E[X^2(t+\tau)X^2(t)] = E\,[X(t+\tau)X(t+\tau)X(t)X(t)]$$

$$R_{yy}(\tau) = R_{xx}(0) + 2R_{xx}(\tau)$$



The output density function

Then, the root mean square value of $Y(t)$is

$$E[Y^2(t)] = R_{yy}(0) = 3\{E[X^2(t)]\}^2 = 3[R_{xx}(0)]^2$$

But $E[Y(t)] = E[X^2(t)] = R_{xx}(0) = \sigma^2$. So the variance of $Y(t)$is

$$\sigma_y^2 = E[Y^2(t)] - \{E[Y(t)\}^2 = 2[R_{xx}(0)]^2 = 2\sigma^4$$

Spectral Power Density

Given a deterministic signal, its Fourier transform (TF) is

$$S(f) = \int_{-\infty}^{+\infty} s(t)\, e^{-j2\pi ft}\, dt$$

The function $S(f)$is also called the spectrum of $s(t)$. Going from the temporal description $s(t)$to the frequency domain $S(f)$, no information about the signal is lost. In other words, $S(f)$forms a complete description of $s(t)$and vice versa. Then $s(t)$can be obtained from $S(f)$by calculating the inverse Fourier transform (IFT). SO :

$$s(t) = \int_{-\infty}^{+\infty} S(f) e^{+j2\pi ft}\, df$$

We treat random processes in the same way as deterministic signals with infinite energy. We define it $x_T(t)$as the sample function $x(t)$, transcribed between $-T\ et\ T$, of the random process $X(t)$. SO :

$$x_T(t) = \begin{cases} x(t), & -T \leq t \leq T \\ 0, & otherwise \end{cases}$$

The transcribed TF of the random process $X(t)$is:

$$X_T(f) = \int_{-T}^{T} x_T(t) e^{-j2\pi ft}\, dt = \int_{-\infty}^{+\infty} x\ \ (t) e^{-j2\pi ft}\, dt$$

The average power of $x_T(t)$is

$$P_{ave} = \frac{1}{2T} \int_{-T}^{T} x_T^2(t) dt$$

Using Parseval's theorem,

$$\int_{-\infty}^{+\infty} x^2\ (t) dt = \int_{-\infty}^{+\infty} |X_T(f)|^2 df$$

The average power of $x_T(t)$is

$$P_T = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{2T} df$$

where the term $|X_T(f)|^2 \,/\, 2T$is the spectral power density of $x_T(t)$. The set mean of $P_T$is given by:

$$E[P_T] = \int_{-\infty}^{+\infty} E\left[\frac{|X_T(f)|^2}{2T}\right] df$$

The spectral power density of the process $X(t)$is defined

$$S_{xx}(f) = \lim_{T \to \infty} E\left[\frac{|X_T(f)|^2}{2T}\right]$$

If $X(t)$is SSL, the spectral power density $S_{xx}(f)$is the TF of the autocorrelation $R_{xx}(\tau)$. SO :

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau$$

Then the spectral power density $S_{xx}(f)$is the Fourier transform of the autocorrelation function $R_{xx}(\tau)$. The latter is therefore the inverse transform of the dsp $S_{xx}(f)$:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f) e^{+j2\pi f\tau} df$$

These two relations are called the ***Wiener-Khinchin relations .*** Note that the dsp is a function of $f$real, positive and even. The autocorrelation function is an τodd function.

Example

Consider the random process $X(t) = A\,cos(\omega_0 t + \Theta)$, where $\Theta$is a random variable uniformly distributed over the interval $[0,\ 2\pi]$, and $A\ \ et\ \omega_0$are constant. Determine the power spectral density of this process.

Solution

As long as $X(t)$it is SSL with an autocorrelation function $R_{xx}(\tau) = (A^2/2)\,cos(2\pi f_0 \tau)$, then:

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau = \int_{-\infty}^{+\infty} \frac{A^2}{2} cos(2\pi f_0 \tau)\, e^{-j2\pi f\tau} d\tau$$

$$= \frac{A^2}{4} [\delta(f - f_0) + \delta(f + f_0)]$$

Inter Power Spectral Densities

Let $X(t)$and be $Y(t)$two jointly stationary random processes in the broad sense. Their power spectral inter-densities

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau$$

And

$$S_{yx}(f) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j2\pi f\tau} d\tau$$

We therefore have, according to the *Wiener-Khinchin relations* and according to:

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

Therefore :

$$S_{yx}(f) = S_{xy}^*(f)$$

Example

Consider the process $Y(t) = X(t - T)$, where $X(t)$is a linear process and SSL with an autocorrelation function $R_{xx}(\tau)$and a power spectral density $S_{xx}(f)$. $T$is a constant. Express the power spectral density $S_{xy}(f)$of the process $Y(t)$in terms of $S_{xx}(f)$.

Solution

The inter-correlation function $R_{xy}(\tau)$is given by:

$$R_{xy}(\tau) = E[X(t + \tau)Y(t)] = E[X(t + \tau)X(t - T)] = R_{xx}(\tau + T)$$

SO :

$$S_{xy}(f) = S_{xx}(f)e^{j2\pi fT}$$

Therefore, the delay time Tappears in the exponent as a phase factored by$2\pi f$

Linear invariant systems

A linear and time-invariant system is characterized by its impulse response $h(t)$, or its transfer function $H(f)$which is the Fourier transform of $h(t)$. SO,

$$H(f) = \int_{-\infty}^{+\infty} h(t)e^{-j2\pi ft}\, dt$$

And

$$h(t) = \int_{-\infty}^{+\infty} H(f)e^{j2\pi ft}\, df$$

if $x(t)$, the input signal applied to the linear system and invariant with respect to time, is deterministic as illustrated in the figure below, the output signal is the convolution of $x(t)$and $h(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)\, d\tau$$

$y(t)$is a sample function of the random process $Y(t)$which corresponds to the sample function of the input random process $X(t)$. The expression of the output in the frequency domain is then:

$$Y(f) = X(f)H(f)$$

where $X(f)$and $Y(f)$are the Fourier transforms of $x(t)$and $y(t)$respectively. The system is feasible provided that the impulse response is causal; SO :

$h(t) = 0 \quad pour \quad t < 0$. In this case, the convolution integral becomes

$$y(t) = \int_{0}^{+\infty} x(t-\tau)h(\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

Stochastic signals

Consider the time-invariant linear system of the previous Figure.

$$Y(t) = h(t) * X(t) = X(t) * h(t)$$

$$= \int_{-\infty}^{+\infty} X(t-\alpha)h(\alpha)\, d\alpha = \int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha$$

The average

The average value of the output process is given by

$$E[Y(t)] = \int_{-\infty}^{+\infty} E[X(t-\alpha)]\, h(\alpha) d\alpha = \int_{-\infty}^{+\infty} m_x(t-\alpha)\, h(\alpha)\, d\alpha$$

where $m_x(t)$is the process average $X(t)$. If $X(t)$is stationary in the broad sense:

$$m_x(t-\alpha) = m_x(t) = constante$$

So the $m_y(t)$process average $Y(t)$is

$$m_y(t) = E[Y(t)] = m_x \int_{-\infty}^{+\infty} h(\alpha)\, d\alpha$$

We know that$\int_{-\infty}^{+\infty} h(\alpha)\, d\alpha = H(0)$

SO :

$$m_y(t) = E[Y(t)] = m_x H(0)$$

The Mean Quadratic Value

$$E[Y^2(t)] = E\left[\iint_{00}^{\infty\infty} X(t-t_1)X(t-t_2)h(t_1)h(t_2)dt_1 dt_2\right]$$

By simplifying this relationship, we will have:

$$E[Y^2(t)] = \iint_{-\infty\,-\infty}^{\infty\,\infty} R_{xx}(t-t_1, t-t_2)h(t_1)\,h(t_2)dt_1dt_2$$

$$E[Y^2(t)] = \iint_{-\infty\,-\infty}^{\infty\,\infty} R_{xx}(t_1, t_2)h(t-t_1)\,h(t-t_2)dt_1dt_2$$

Assuming that $X(t)$is stationary in the broad sense and after a change of variable

$\alpha = t - t_1 \qquad \beta = t - t_2$, SO :

$$E[Y^2(t)] = \iint_{-\infty\,-\infty}^{\infty\,\infty} R_{xx}(\alpha - \beta)h(\alpha)\,h(\beta)d\alpha d\beta$$

Which is independent of time

The Intercorrelation Function Between Input and Output

We consider that the input process $X(t)$is stationary in the broad sense. The intercorrelation function between input and output is:

$$R_{yx}(t+\tau, t) = E[Y(t+\tau)X^*(t)]$$

Using the relationship $Y(t) = X(t) * h(t)$and after changing variables, the intercorrelation function can be written as follows:

$$R_{yx}(t+\tau, t) = \int_{-\infty}^{+\infty} R_{xx}(\tau - \alpha)h(\alpha)d\alpha = R_{xx}(\tau) * h(\tau)$$

We see that this result does not depend on t, and therefore $R_{yx}(t+\tau) = R_{yx}(\tau)$. we can also show that:

$$R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau)$$

We directly calculate the power spectral interdensity $S_{yx}(f)$:

$$S_{yx}(f) = S_{xx}(f) \cdot H(f)$$

And

$$S_{xy}(f) = S_{xx}(f) \cdot H^*(f)$$

The Autocorrelation Function and the Output Spectrum

$$R_{yy}(t+\tau,t) = E[Y(t+\tau)Y(t)]$$

We have

$$Y(t+\tau) = \int_{-\infty}^{+\infty} X(t+\tau-\alpha)h(\alpha)\,d\alpha$$

And

$$Y(t) = \int_{-\infty}^{+\infty} X(t-\beta)h(\beta)\,d\beta$$

By substitution of $Y(t+\tau)$and $Y(t)$in $R_{yy}(t+\tau,t)$and by change of variables: $\alpha = -\beta$, we find:

$$R_{yy}(\tau) = E\left[\iint_{-\infty-\infty}^{\infty\infty} X(t+\tau-\alpha)h(\alpha)X(t-\beta)h(\beta)d\alpha d\beta\right]$$

$$= \iint_{-\infty}^{+\infty} E[X(t+\tau-\alpha)X(t-\beta)]h(\alpha)h(\beta)\,d\alpha d\beta.$$

$$= \iint_{-\infty}^{+\infty} R_{xx}(\tau-\alpha+\beta)h(\alpha)h(\beta)d\alpha d\beta = R_{xx}(\tau) * h(\tau) * h(-\tau)$$

$$R_{yy}(\tau) = R_{yx}(\tau) * h(-\tau) = R_{xy}(\tau) * h(\tau) = R_{xx}(\tau) * h(\tau) * h(-\tau)$$

By calculating the Fourier transform of $R_{yy}(\tau)$, we find the spectral density of the output$S_{yy}(f)$

$$S_{yy}(f) = S_{xx}(f) \cdot |H(f)|^2$$

Example

White noise with an autocorrelation function $R_{xx}(\tau) = (N_0/2)\delta(\tau)$applied to a filter with an impulsive response:

$$h(t) = \begin{cases} \alpha\, e^{-\alpha t}, & t \geq \quad and \quad \alpha > 0 \\ 0, & t < 0 \end{cases}$$

Determine the autocorrelation function $R_{yy}(\tau)$of the output process?

Solution

The problem can be solved by two methods. We can directly calculate the convolution intergral $R_{yy}(\tau)$or the power spectral density $S_{yy}(f)$as a function of $S_{xx}(f)$, then we calculate its inverse Fourier Transform of $S_{yy}(f)$.

Method 1

For $\tau < 0$, we have:

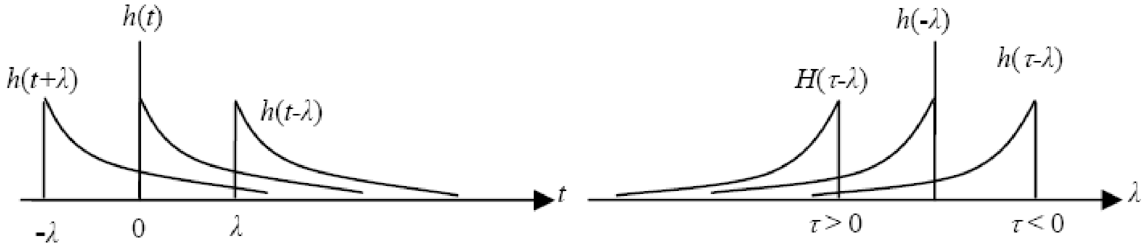
$$h(\tau) * h(-\tau) = \int_{-\infty}^{\tau} \alpha e^{-\alpha(\tau-\lambda)} \alpha\, e^{+\alpha\lambda} d\lambda = \alpha^2\, e^{-\alpha\tau} \int_{-\infty}^{\tau} e^{2\alpha\lambda} d\lambda = \frac{\alpha}{2} e^{\alpha\tau}$$

For $\tau > 0$, we have:

$$h(\tau) * h(-\tau) = \int_{-\infty}^{0} \alpha e^{-\alpha(\tau-\lambda)} \alpha e^{\alpha\lambda} d\lambda = \frac{\alpha}{2}\, e^{-\alpha\tau}$$

SO

$$g(\tau) = h(\tau) * h(-\tau) = \begin{cases} \frac{\alpha}{2} e^{\alpha\tau}, & \tau \leq 0 \\ \frac{\alpha}{2}\, e^{-\alpha\tau}, & \tau \geq 0 \end{cases}$$



The impulse response with τparameter

SO

$$R_{yy}(\tau) = R_{xx}(\tau) * g(\tau) = \begin{cases} \dfrac{N_0\alpha}{4} e^{\alpha\tau}, & \tau \leq 0 \\ \dfrac{N_0\alpha}{4} e^{-\alpha\tau}, & \tau \geq 0 \end{cases}$$

$$R_{yy}(\tau) = \frac{N_0\alpha}{4} e^{-\alpha|\tau|}$$

Method 2

From the relationship of the output dsp, $S_{yy}(f)$we must first calculate the TF $H(f)$of the impulse response $h(t)$. SO :

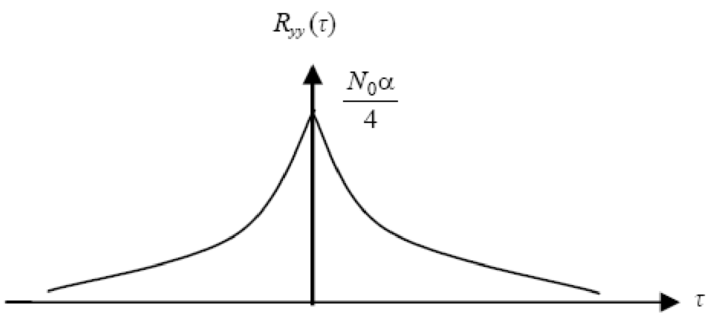
$$H(f) = \int_0^\infty \alpha e^{-\alpha t} e^{-j2\pi f t} dt = \alpha \int_0^\infty e^{-(j2\pi f+\alpha)t} dt = \frac{\alpha}{j2\pi f + \alpha}$$

$$|H(f)|^2 = \frac{\alpha^2}{4\pi^2 f^2 + \alpha^2}$$

While the spectral density of the output is:

$$S_{yy}(f) = S_{xx}(f)|H(f)|^2 = \frac{N_0\alpha}{4}\frac{\alpha^2}{\omega^2 + \alpha^2}$$

Or $\omega = 2\pi f$. By calculating the inverse Fourier transform of $S_{yy}(f)$, we obtain the autocorrelation function$R_{yy}(\tau)$



The autocorrelation function of $Y(t)$

$$R_{yy}(\tau) = \frac{N_0\alpha}{4} e^{-\alpha|\tau|}$$

White noise

White noise $b(t)$is a stationary random process (SSL) at TC or TD, generally centered, whose DSP $S_{bb}(f)$is constant over the entire frequency axis (the name "white" therefore refers to white light whose power is distributed uniformly over all optical frequencies). Due to the definition of the DSP, white noise is therefore characterized by its impulsive autocorrelation function $R_{bb}(\tau)$.

Variance White Noiseσ^2

The autocorrelation function

at TC: $R_{bb}(\tau) = \sigma^2\delta(\tau)$at TD:$R_{bb}(k) = \sigma^2\delta_{k,0}$

the DSP

at TC: $S_{bb}(f) = TF[R_{bb}(\tau)] = \sigma^2$at TD:$S_{bb}(f) = TF[R_{bb}(k)] = \sigma^2$

Other Definitions:

Weak sense: white noise is a series of uncorrelated VAs (uncorrelated realizations)

Strong meaning: white noise is a series of independent VAs (independent realizations)

The term "whiteness" comes from the analogy with white light and reflects the fact that all frequencies are present in white noise with the same power.

A white noise at TD is achievable in practice whereas a white noise at TC is not because its power (which is equal to its autocorrelation function at 0) is infinite (Dirac).

Distribution law: a white process can have any distribution law: normal, uniform, etc. ….

Ergodicity

A random process $X(t)$can be seen as a multitude of trajectories corresponding to as many realizations from experience to identity. However, in a large number of practical cases, only one realization of the process is accessible to the measurement

A stationary random process $X(t)$is ergodic if all its statistics can be determined from a single realization; that is to say if its average and its autocorrelation function can be obtained by performing a temporal average on a single trajectory (a single realization) of infinite duration.

Precisely, we speak of ergodicity in the sense of the average and in the sense of the autocorrelation function.

Ergodicity in the sense of the mean

A random process $X(t)$is ergodic in the sense of mean if:

$$E[X(t)] = \langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)dt$$

The necessary and sufficient condition under which the process $X(t)$is ergodic in the sense of the average is:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(\tau)d\tau = m_x^2$$

Where $m_x = E[X(t)]$is the average of$X(t)$

Ergodicity in the sense of the autocorrelation function

A random process $X(t)$is ergodic in the sense of the autocorrelation function if

$$R_{xx}(\tau) = \langle x(t+\tau)x(t) \rangle$$

As $\langle x(t+\tau)x(t) \rangle$ denotes the time average of the autocorrelation function of the realization $x(t)$and is defined by:

$$\langle x(t+\tau)x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau)x(t)\, dt$$

The necessary and sufficient condition for ergodicity in the sense of the autocorrelation function is that the random variables $X(t+\tau)X(t)$and $X(t+\tau+\alpha)X(t+\alpha)$become uncorrelated for each $\tau$when αtends to infinity.

Example :

Consider a random process $X(t) = A\cos(2\pi f_c t + \Theta)$, where $A$and $f_c$are constant, and $\Theta$is a random variable uniformly distributed over the interval $[0, 2\pi]$.

Solution

We have already calculated for this process:

$E[X(t)] = 0$, And$R_{xx}(\tau) = \left(\frac{A^2}{2}\right) cos(2\pi f_c \tau)$

Either the completion of the process$x(t) = Acos(2\pi f_c t + \theta)$

The temporal average$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Acos(2\pi f_c t + \theta)dt = 0$

And$\langle x(t+\tau)x(t)\rangle = \lim_{T\to\infty} \frac{A^2}{2T}\int_{-T}^{T} cos(2\pi f_c (t+\tau) + \theta)cos(2\pi f_c t + \theta)dt$

$$= \left(\frac{A^2}{2}\right) cos(2\pi f_c \tau)$$

Then, the process is ergodic in the sense of the mean and in the sense of the autocorrelation function.

1st Order Distribution Function

Consider $X(t)$a stationary random process. We define the random process $Y(t)$as follows:

$$Y(t) = \begin{cases} 1, & X(t) \leq x_t \\ 0, & X(t) > x_t \end{cases}$$

We say that the random process $X(t)$is ergodic in the sense of the 1st order distribution [if] :

$$F_X(x;t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)dt$$

Where $F_X(x;t) = P[X(t) \leq x(t)]$and $y(t)$is a sample function of the process $Y(t)$. The necessary and sufficient condition under which the process is ergodic in the sense of the 1st order distribution [is] that $X(t+\tau)$and $X(t)$becomes statistically independent when τtends to infinity.

Ergodicity in the Sense of Power Spectral Density

The stationary process in the broad sense (SSL) $X(t)$is ergodic in the sense of power spectral density if, for any sample function $x(t)$,

$$S_{xx}(f) = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} x(t) e^{-j2\pi ft} \, dt \right|^2$$

Except for the set of sample functions which occur with zero probability.

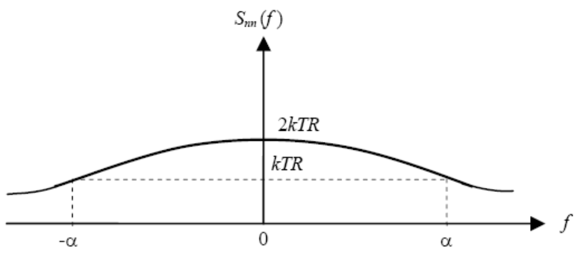
Examples of random processes:

Thermal noise

The electrical noise that appears from the random movement of electrons in conductors is called thermal noise. It can be shown that the power spectral density of the thermal noise voltage across a resistor is given by r:

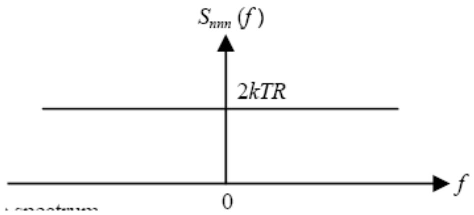
$$S_{nn}(f) = 2kTR \frac{\alpha^2}{\alpha^2 + \omega^2}$$

where $k = 1.38 \times 10^{-23}\ J/K$is the Boltzmann constant, and $T$is the absolute temperature in K. Figure 2.10 illustrates the shape of the power spectral density of thermal noise.



spectral density of thermal noise

However, $\alpha$is of the order of $10^{14}\ rad/s(\ 10^{13}\ Hz = \ 10^{4}\ GHz$which is higher than most frequencies used in electrical circuits. Then $(\alpha^2 + \omega^2)/\alpha^2 \to 1$thermal noise is considered as a white noise process with a flat spectrum of value $2kTR\ V^2/Hz$as is shown in Figure 2.11.



white noise spectrum

Additionally, since the number of electrons in a resistor is very large with statistically independent random movements, from the central limit theorem, the thermal noise is modeled as Gaussian with zero mean. As a result, the thermal noise voltage is a white Gaussian process with zero mean. The resistance can be modeled by the Thévenin equivalent circuit, which consists of a non-noise resistance in series with a voltage noise source. Figure 2.12 (a) illustrates this circuit. The mean square value of this noisy source is:

$$E[V_n^2(t)] = 4kTR$$

We can also model the noisy resistance by an equivalent Norton circuit, which is made up of a non-noise resistance in parallel with a current noise source as shown in Figure 2.12 (b). The root mean square value of this source is:

$$E[I_n^2(t)] = 4kTG$$

where $G = 1/R$is an admittance. The spectral density of the noise voltage source and the noise current source are respectively:

$$S_{v_n v_n}(f) = 2kTR \quad V^2/Hz$$

$$S_{i_n i_n}(f) = 2kTG \quad A^2/Hz$$

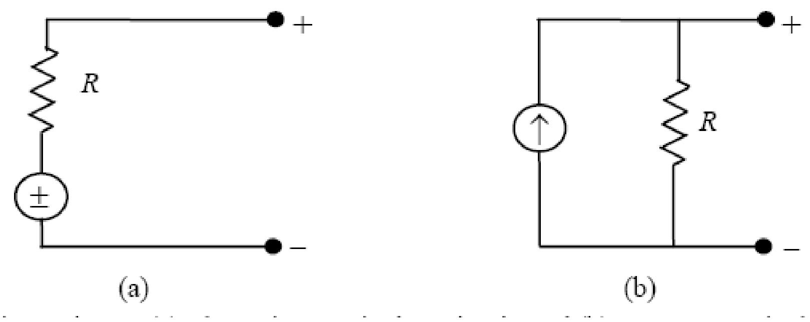


Fig.2.12 Noisy resistance: (a) Thévnin’s equivalent circuit

(b) Norton's equivalent circuit

Nyquist's theorem

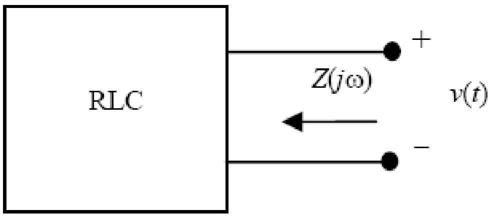
Consider a passive RLC network as shown in Figure 2.13. the voltage across the network is $v(t)$and $Z(j\omega)$is the impedance. Then, the power spectral density of the open circuit noise voltage due to all thermal noise sources is given by:

$$S_{v_n v_n}(f) = 2kT\Re\{Z(j\omega)\}$$

Or, the power spectral density of the short circuit noise current is given by:

$$S_{i_n i_n}(f) = 2kT\Re\{Y(j\omega)\}$$

where $Y(j\omega) = 1/Z(j\omega)$is the admittance of the network input and$\omega = 2\pi f$

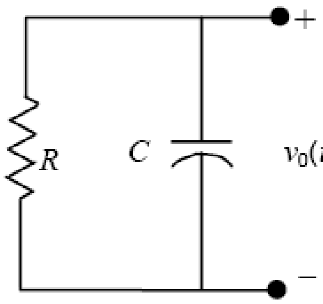


The passive RLC network

Example

Determine the power spectral density of the voltage $v(t)$across the RC network in Figure 2.14, due to the thermal noise generated in R, using:

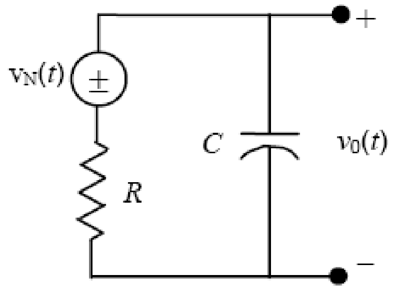
a) Thévenin's equivalent circuit
b) Norton's equivalent circuit
c) Nyquist's theorem



The RC network

Solution

Thévenin equivalent circuit



Thévenin equivalent circuit

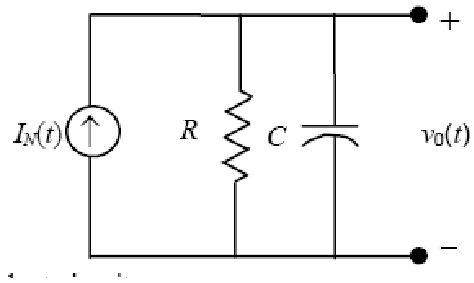
(a) Using the Thévenin equivalent circuit, the transfer function of the noise source (by applying the voltage divider) is:

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

SO :

$$S_{v_0 v_0}(\omega) = S_{v_n v_n}(\omega)|H(j\omega)|^2 = \frac{2kTR}{1 + (\omega RC)^2}$$

(b) Using Norton's equivalent circuit, the resulting circuit is shown in Figure 2.15.



Norton equivalent circuit

The transfer function in this case is:

$$H(j\omega) = \frac{\frac{R}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{R}{1 + j\omega RC}$$

The power spectral density of the output voltage is:

$$S_{v_0v_0}(\omega) = S_{i_ni_n}(\omega)|H(j\omega)|^2 = \frac{2kT}{R}\frac{R^2}{1+(\omega RC)^2} = \frac{2kTR}{1+(\omega RC)^2}$$

(c) The impedance seen at the network terminals is:

$$Z(j\omega) = \frac{\frac{R}{j\omega C}}{R+\frac{1}{j\omega C}} = \frac{R}{1+j\omega RC} = \frac{R}{1+(\omega RC\omega)^2} - j\,\frac{\omega RC}{1+(\omega RC\omega)^2}$$

According to Nyquist's theorem, the power spectral density of the noise voltage source is:

$$S_{v_0v_0}(\omega) = 2kT\Re e\{Z(j\omega)\} = \frac{2kTR}{1+(\omega RC)^2}$$

We notice that the three methods of (a), (b) and (c) lead to the same result.

Generally, the power spectral density of the white noise process is denoted by:

$$S_{nn}(f) = \frac{N_0}{2}, \quad -\infty < f < \infty$$

The autocorrelation function is then:

$$R_{nn}(\tau) = \frac{N_0}{2}\delta(\tau)$$

Since the bandwidth of real systems is finite, DSP $S_{nn}(f)$over a finite frequency band leads to a finite average power.

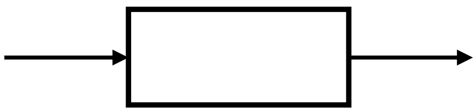
CHAPITRE VII

Discrete Random Processes

Introduction

In this Chapter, we consider another class of random processes; namely discrete-time stochastic processes. A discrete random process can be a uniformly sampled version of a continuous random process. It is a collection of a real or complex set of discrete sequences of time. The discrete sequence is also called realizations and denoted by $X(n)$. $X(n)$represents a random variable. The sequence $X(n), X(n-1), \dots, X(n-M+1)$consists of the next observation and the previous observations $n-1, n-2, \dots, n-M+1$. As a result, several discrete random processes are approximated by a parametric model. Power spectral density is a function of a parametric model. Then, the choice of a model and the estimation of the model parameters are necessary. These approaches are called: parametric estimation.

If $U(n)$is an input sequence and $X(n)$the output sequence, then the general recursive model of this system is given by the equation:



$$X(n) = -\sum_{k=1}^{p} a(k)X(n-k) + \sum_{k=0}^{q} b(k)U(n-k)$$

The calculation of the spectrum using this parametric model is called parametric spectral estimation. In fact, parametric spectral estimation is a very broad field, we are not going to address this problem. However, we will introduce

- The autoregressive process (**AR**);
- The moving average process (**MA**),
- The autoregressive moving average (**ARMA**) process

Before discussing these models, it is worth recalling the Wiener-Khinchin relations of the power spectral density and the autocorrelation function for a discrete random process.

Consider $r_{xx}(k)$the autocorrelation function of a discrete random process. Then the Fourier transform of $r_{xx}(k)$is the power spectral density$S_{xx}(\omega)$

$$S_{xx}(\omega) = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-j\omega k}, \qquad |\omega| < \pi$$

where $\omega = 2\pi f$ is the angular frequency.

Moreover,

$$S_{xx}(\omega + 2\ell\pi) = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-j(\omega+2\ell\pi)k} = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-j\omega k}\, e^{-j2\ell\pi k} = S_{xx}(\omega)$$

$e^{-j2\ell\pi k} = 1$

Then, the power spectral density is a periodic function of period 2π.

$$r_{xx}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)e^{-j\omega k}\, d\omega$$

The root mean square value which represents the average power of the process is:

$$r_{xx}(0) = E[|X(n)|^2] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)d\omega = \int_{-f/2}^{f/2} S_{xx}(f)df$$

$\omega = 2\pi f$

The power spectral density is real because:

$$r_{xx}(-k) = r_{xx}(n, n-k) = E[X(n)X^*(n-k)]$$

$$= E[X^*(n-k)X(n)] = r_{xx}^*(k)$$

By analogy, the interpower spectral density is defined by:

$$S_{xy}(\omega) = \sum_{k=-\infty}^{\infty} r_{xy}(k)e^{-j\omega k}$$

For greater clarity, we consider the following discrete linear system:

Fig. 7.1 discrete linear system

With $h(n)$ the impulse response of the system.

$$r_{xy}(k) = h(k) * r_{xx}(k) = \sum_{\ell=-\infty}^{\infty} h(\ell) r_{xx}(k-\ell) = r_{xx}(k) * h(k) = \sum_{\ell=-\infty}^{\infty} r_{xx}(\ell) h(k-\ell)$$

And

$$r_{yx}(k) = h^*(-k) * r_{xx}(k) = \sum_{\ell=-\infty}^{\infty} h^*(l) r_{xx}(k-l)$$

The autocorrelation function of the output process is:

$$r_{yy}(k) = h(k) * r_{yx}(k) = h(k) * h^*(-k) * r_{xx}(k)$$

The corresponding spectral densities are therefore:

$$S_{xy}(\omega) = \sum_{k=-\infty}^{\infty} r_{xy}(k) e^{-j\omega k}$$

And

$$S_{yx}(\omega) = \sum_{k=-\infty}^{\infty} r_{yx}(k) e^{-j\omega k}$$

With the transformation into Z, we will have:

$$S_{xy}(Z) = H(Z) S_{xx}(Z)$$

And

$$S_{yx}(Z) = H^*\left(\frac{1}{Z^*}\right) S_{xx}(Z)$$

And

$$S_{yy}(Z) = H(Z) H^*\left(\frac{1}{Z^*}\right) S_{xx}(Z)$$

where, $H(Z)$is the transform $Z$of $h(n)$, also denoted by $Z\{h(n)\}$, and is given by:

$$H(Z) = \sum_{n=-\infty}^{\infty} h(n) Z^{-n}$$

The frequency response $H(e^{j\omega})$can be deduced from $H(Z)$evaluated on the unit circle of the plane $Z(Z = e^{j\omega})$. For $h(n)$real, $H^*\left(\frac{1}{Z^*}\right) = H\left(\frac{1}{Z}\right)$and the spectral density of the output is given by:

$$S_{yy}(\omega) = \left|H\left(e^{j\omega}\right)\right|^2 S_{xx}(\omega)$$

Example 4.1

Consider the system in Figure 7.1. determine the power spectral density of the output, if the input $X(n)$is a stationary white noise process

Solution :

The white noise autocorrelation function is:

$$r_{xx}(n, n+k) = r_{xx}(k) = \sigma_n^2 \delta(k)$$

With

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

The power spectral density $S_{xx}(f)$is

$$S_{xx}(\omega) = \sigma_n^2$$

Then, the spectral density of the output is:

$$S_{yy}(\omega) = S_{xx}(\omega)\left|H\left(e^{j\omega}\right)\right|^2 = \sigma_n^2\left|H\left(e^{j\omega}\right)\right|^2$$

2. AR, MA AND ARMA random processes

The power spectral density of a random process plays an essential role in spectral estimation. It provides important information on the structure of the process. Such information can be used in several applications, such as prediction, modeling or filtering of the observed signal. There are two spectral estimation methods:

1. Non-parametric spectral estimation: which is based on the autocorrelation function (second order statistics of the process). We therefore estimate a quantity.
2. Parametric spectral estimation: which is based on a parametric model of the power spectral density. We must therefore first choose the model, then estimate the parameters.

We consider the following system:

Such that the input is a white noise process. The output is described by a parametric model; the power spectral density of the output is expressed as a function of the parameters. Therefore, it is imperative to choose a parametric model and estimate its parameters. The most used models are:

- The AR (autoregressive) model
- The MA (moving average) model

- The ARMA (autoregressive moving average) model

4.2.1 Autoregressive (AR) Model

An AR process is represented by the following equation:

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + e(n) = \sum_{k=1}^{p} \omega_k X(n-k) + e(n)$$

where $X(n)$is the actual observed random sequence,

$a_k,\ \ k = 1,2,\dots,p$, are constants called parameters, such that $\omega_k = -a_k$

$e(n)$is the sequence of independent and identically distributed random variables (iid), white Gaussian noise process of zero mean with unknown varianceσ_n^2

$p$is the order of the filter.

The sequence $X(n)$is called an autoregressive model of order p, and ,denoted by $AR(p)$. The term autoregressive arises from the fact that $X(n)$, the present value of the process is given by

$$X(n) = -a_1 X(n-1) - a_2 X(n-2) - \dots - a_p X(n-p) + e(n)$$

It is a linear combination of $X(n-1),\ X(n-2),\dots,X(n-p)$, the previous values of the process, and the term $e(n)$. The transform $Z$is given by:

$$X(Z) = -\sum_{k=1}^{p} a_k Z^{-k} X(Z) + E(Z)$$

$\Longrightarrow$ $$X(Z)\left(1 + \sum_{k=1}^{p} a_k Z^{-k}\right) = E(Z)$$

$$X(Z)\left(1 + a_1 Z^{-1} + a_2 Z^{-2} + \dots + a_p Z^{-p}\right) = E(Z)$$

where is the $X(Z)$de $X(n)$transform $Z$and is the $E(Z)$de $e(n)$transform Z, and

$$TZ(X(n-k)) = Z^{-k} X(Z),\ \ k = 1,2,\dots,p$$

The transfer function of the filter, $H(Z)$, is therefore:

$$H(Z) = \frac{X(Z)}{E(Z)} = \frac{1}{1 + \sum_{k=1}^{p} a_k Z^{-k}}$$

$Z = e^{j\omega}$ and so $H\left(e^{j\omega}\right) = \frac{1}{1+\sum_{k=1}^{p} a_k e^{-j\omega k}}$

Let us recall that the spectral density of the output $X(n)$is:

$$S_{xx}(f) = |H(f)|^2 S_{ee}(f)$$

$e(n)$is centered and variance Gaussian white noise σ_n^2 , and$S_{ee}(f) = \sigma_n^2$

SO :

$$S_{xx}(f) = \frac{\sigma_n^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi fk}\right|^2}$$

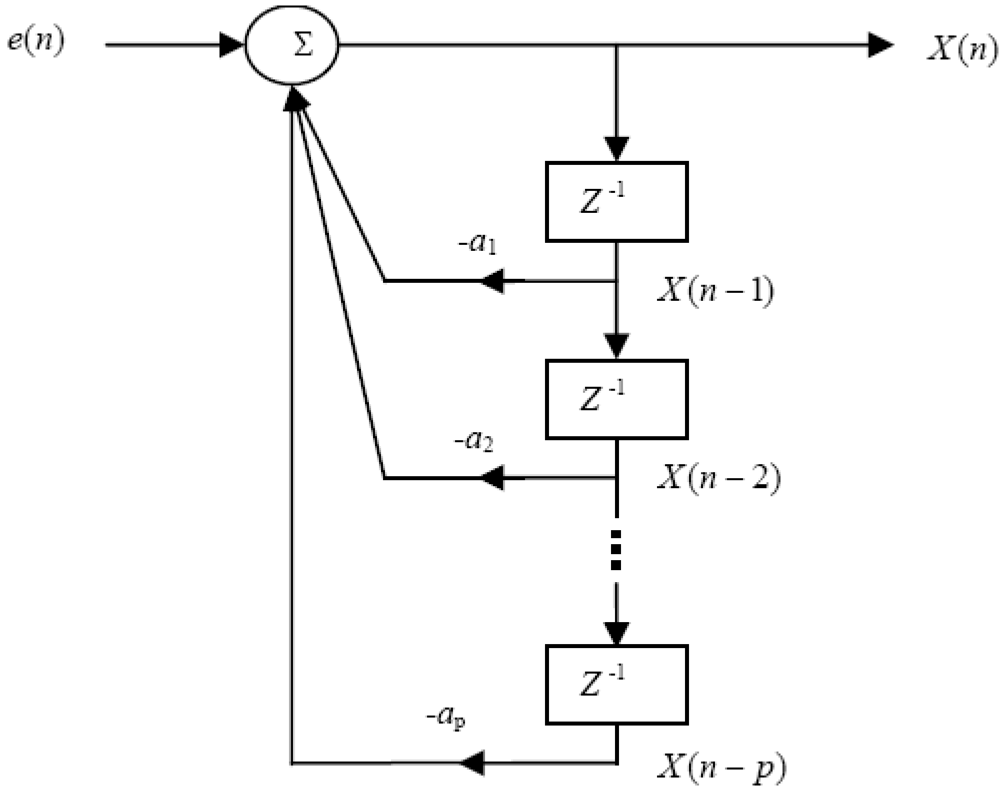


Fig. 7.2 . Creating an $AR(p)$all-pole filter

To study the $X(n)$model process $AR(p)$, it is necessary to determine the mean, the autocorrelation function, the correlation coefficients and the power spectral density which will be a function of the model parameters. The process is assumed to be stationary.

model $AR(p)$averageE$[X(n)]$

The average of $X(n)$is given by:

$$E[X(n)] = m_x = E\left[-\sum_{k=1}^{p} a_k X(n-k) + e(n)\right] = -\sum_{k=1}^{p} E[X(n-p)]$$

With $E[e(n)] = 0$. We start with$p = 1$

Process$AR(1)$

For $p = 1$, the order 1 process is:

$$X(n) = -a_1 X(n-1) + e(n)$$

The first order average is then:

$$E[X(n)] = m_x = -E[a_1 X(n-1)] = -a_1 m_x$$

If $a_1 \neq 0$ then $m_x = 0$

Process $AR(1)$ Variance σ_x^2

$$\sigma_x^2 = E[X(n)X^*(n)] = E\{[-a_1 X(n-1) + e(n)][-a_1 X^*(n-1) + e^*(n)]\}$$

$$= \frac{\sigma_n^2}{1 - a_1^2}$$

As long as the initial condition is assumed to be Gaussian, $E[X(0)] = 0$uncorrelated and independent of the Gaussian white noise process. SO

$$E[X(n-1)e(n)] = 0$$

$$\sigma_x^2 = a_1^2 \sigma_x^2 + \sigma_n^2$$

The variance must be finite and positive, then $|a_1^2| < 1$,$-1 < a_1 < 1$

Process $AR(1)$ autocorrelation function $r_{xx}(k)$

$$r_{xx}(k) = E[X(n)X(n-1)]$$

$$r_{xx}(k) = (-1)^k a_1^k r_{xx}(0)$$

Autocorrelation coefficient of the process $AR(1)$

$$\rho_k = \frac{r_{xx}(k)}{r_{xx}(0)} = (-1)^k a_1^k$$

Process power spectral density $AR(1)$

$$S_{xx}(f) = |H(f)|^2 S_{ee}(f)$$

Such that $S_{ee}(f)$is the power spectral density of the noise$S_{ee}(f) = \sigma_n^2$

The transfer function is given by :

$$H(e^{j\omega}) = \frac{1}{1 + a_1 e^{-j\omega}}, \quad |\omega| < \pi, \; Z = e^{j\omega}$$

And

$$|H(e^{j\omega})|^2 = \frac{1}{1 + 2a_1 cos2\pi f + a_1^2}, \quad |f| < \frac{1}{2}, \; \omega = 2\pi f$$

Then the power spectral density of the process$AR(1)$

$$S_{xx}(f) = \frac{\sigma_n^2}{1 + 2a_1 cos2\pi f + a_1^2} = \frac{\sigma_x^2(1 - a_1^2)}{1 + 2a_1 cos2\pi f + a_1^2}$$

Where $\sigma_n^2$is deduced from the relation ofσ_x^2

Process$AR(p)$

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + e(n)$$

With an average

$$E[X(n)] = m_x = 0 \quad for \quad \sum_{k=1}^{p} a_k \neq 1$$

and a variance

$$\sigma_x^2 = E[X(n)X^*(n)] = -\sum_{k=1}^{n} a_k r_{xx}(k) + \sigma_n^2$$

Process power spectral density $AR(p)$

$$S_{xx}(f) = \frac{\sigma_n^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi f k}\right|^2}$$

To determine, it is necessary to estimate the $S_{xx}(f)$process parameters .a_k

4.2.2 MA (Moving Average) process

$$X(n) = \sum_{k=0}^{q} b_k e(n-k)$$

Where $b_0, b_1, \ldots, b_q$ constants are called the MA parameters, such that $\sum_{k=0}^{q} b_k = 1$and $e(n)$is the input (white noise process).

The Transfer Function

$$H(Z) = 1 + b_1 Z^{-1} + b_2 Z^{-2} + \ldots + b_q Z^{-q}$$

$b_0 = 1$

The transfer function is then:

$$H\left(e^{j2\pi f}\right) = 1 + \sum_{k=1}^{q} b_k e^{-j2\pi fk}$$

The average

$$E[X(n)] = m_x = E\left[e(n) + \sum_{k=1}^{q} b_k e(n-k)\right] = 0$$

The variance

$$\sigma_x^2 = \sigma_n^2\left(1 + \sum_{i=1}^{q} b_i^2\right)$$

The autocorrelation function

$$r_{xx}(k) = E\left\{\left[e(n) + \sum_{i=1}^{q} b_i e(n-i)\right]\left[e(n-k) + \sum_{j=1}^{q} b_i e(n-k-j)\right]\right\}$$

$$= r_{ee}(k) + \sum_{j=1}^{q} b_j r_{ee}(k+j) + \sum_{i=1}^{q} b_i r_{ee}(k-i) + \sum_{i=1}^{q}\sum_{j=1}^{q} b_i b_j r_{ee}(k+j-i)$$

$$= \begin{cases} \sigma_n^2\left(1 + \sum_{j=1}^{q} b_j^2\right) & , \quad k = 0 \\ \sigma_n^2\left(b_1 + \sum_{j=2}^{q} b_j b_{j-1}\right), & k = 1 \\ \sigma_n^2\left(b_1 + \sum_{j=3}^{q} b_j b_{j-2}\right), & k = 2 \end{cases}$$

Extera, until $k = q$. The general form can therefore be deduced as follows:

$$r_{xx}(k) = \begin{cases} \sigma_n^2\left(b_k + \sum_{j=k+1}^{q} b_j b_{j-k}\right), & k < q \\ \sigma_n^2 b_q, & k = q \\ 0, & k > q \end{cases}$$

Power Spectral Density

$$S_{xx}(f) = \sigma_n^2 \left|1 + \sum_{k=1}^{q} b_k e^{-j2\pi fk}\right|^2, \quad |f| < \frac{1}{2}$$

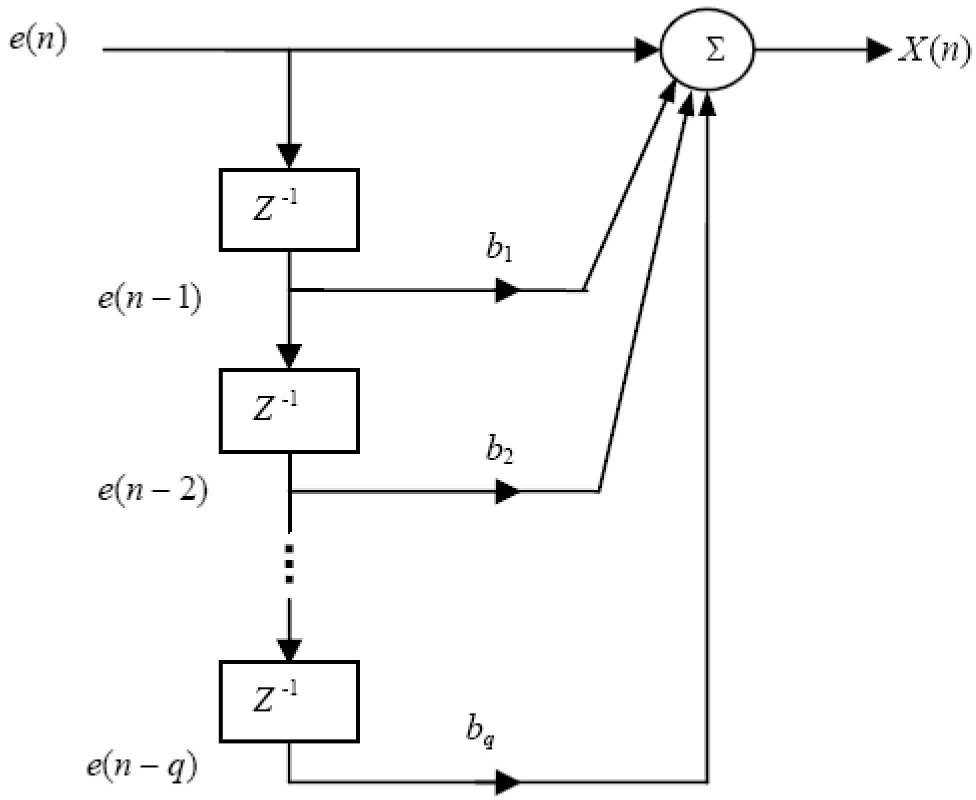


Fig. 7.3. Creation of an order moving average filter$q$ MAall zeros

4.2.3 ARMA process

$$X(n) + a_1X(n-1) + \ \dots \ + a_pX(n-p) = e(n) + b_1e(n-1) + \ \dots + b_qe(n-q)$$

$$X(n) = -\sum_{k=1}^{p} a_kX(n-k) + e(n) + \sum_{l=1}^{q} b_le(n-l)$$

By calculating the Z transform and from the impulse response

$$H(Z) = \frac{1 + \sum_{l=1}^{q} b_lZ^{-l}}{1 + \sum_{k=1}^{p} a_kZ^{-k}}$$

Power Spectral Density

$$S_{xx}(f) = \sigma_n^2 \frac{\left|1 + \sum_{l=1}^{q} b_l e^{-j2\pi fl}\right|^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi fk}\right|^2}, \quad |f| < \frac{1}{2}$$

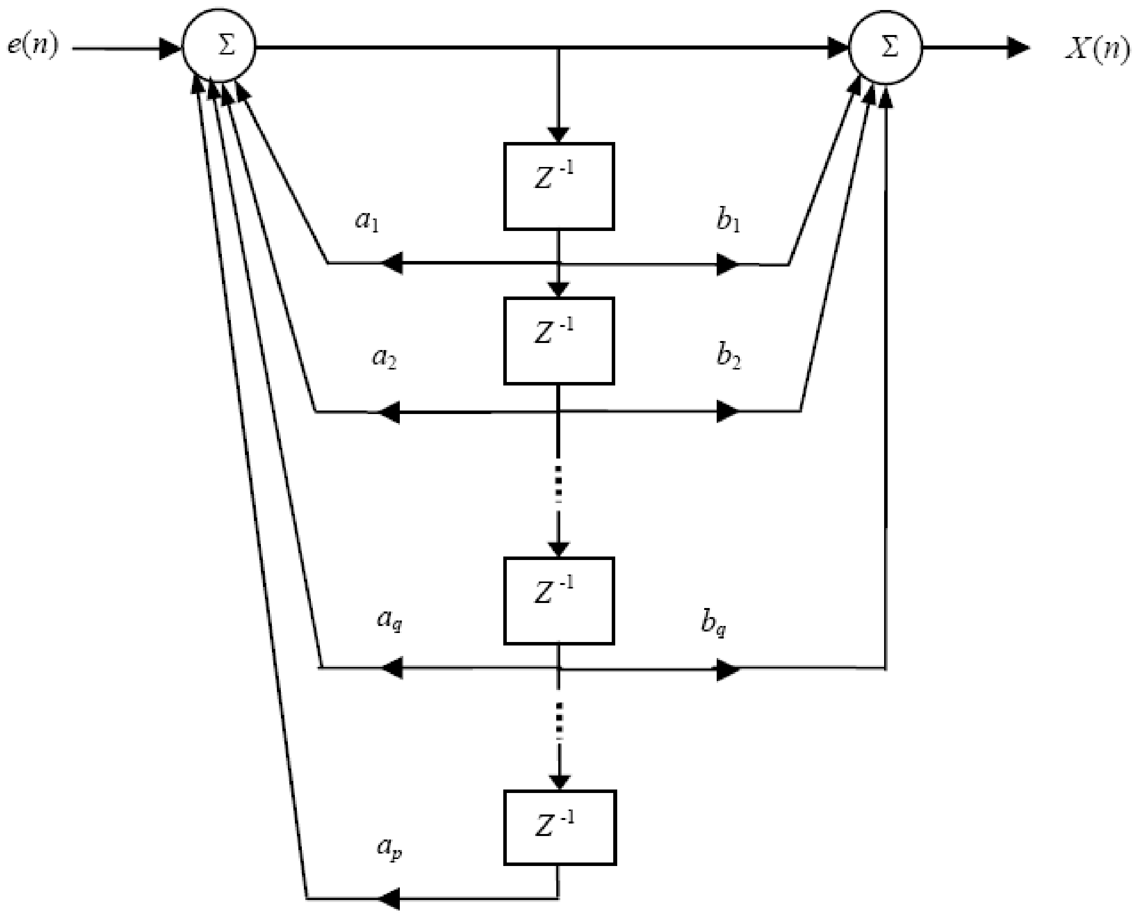


Fig. 7.4. Creation of an Autoregressive Moving Average order filter(p,q) $ARMA$with $p > q$

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