Nonlinear Systems and Control
Lecture # 1
Introduction

Nonlinear State Model

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{aligned}$$

$\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t

u_1, u_2, ..., u_p are input variables

x_1, x_2, ..., x_n the state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

$$\dot{x} = f(t,x,u)$$

$$\begin{aligned} \dot{x} &= f(t,x,u) \\ y &= h(t,x,u) \end{aligned}$$

x is the state, u is the input
y is the output (q-dimensional vector)

Special Cases:

Linear systems:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

Unforced state equation:

$$\dot{x} = f(t,x)$$

Results from $\dot{x} = f(t,x,u)$ with $u = \gamma(t,x)$

Autonomous System:

$$\dot{x} = f(x)$$

Time-Invariant System:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

A time-invariant state model has a time-invariance property with respect to shifting the initial time from t_0 to $t_0 + a$, provided the input waveform is applied from $t_0 + a$ rather than t_0

Existence and Uniqueness of Solutions

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x over the domain of interest

$f(t, x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities

$f(t, x)$ is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0, r) = \{x \in R^n \mid \|x - x_0\| < r\}$ where $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\|, \ \ L > 0$$

A function $f(t, x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$

When $n = 1$ and f depends only on x

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

On a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L

Any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point

A discontinuous function is not locally Lipschitz at the points of discontinuity

The function $f(x) = x^{1/3}$ is not locally Lipschitz at $x = 0$ since

$$f'(x) = (1/3)x^{-2/3} \rightarrow \infty \text{ a } x \rightarrow 0$$

On the other hand, if $f'(x)$ is continuous at a point x_0 then $f(x)$ is locally Lipschitz at the same point because continuity of $f'(x)$ ensures that $|f'(x)|$ is bounded by a constant k in a neighborhood of x_0 ; which implies that $f(x)$ satisfies the Lipschitz condition $L = k$

More generally, if for $t \in J \subset R$ and x in a domain $D \subset R^n$, $f(t, x)$ and its partial derivatives $\partial f_i / \partial x_j$ are continuous, then $f(t, x)$ is locally Lipschitz in x on D

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$

Without the local Lipschitz condition, we cannot ensure uniqueness of the solution. For example, $\dot{x} = x^{1/3}$ has $x(t) = (2t/3)^{3/2}$ and $x(t) \equiv 0$ as two different solutions when the initial state is $x(0) = 0$

The lemma is a local result because it guarantees existence and uniqueness of the solution over an interval $[t_0, t_0 + \delta]$, but this interval might not include a given interval $[t_0, t_1]$. Indeed the solution may cease to exist after some time

Example:

$$\dot{x} = -x^2$$

$f(x) = -x^2$ is locally Lipschitz for all x

$$x(0) = -1 \;\Rightarrow\; x(t) = \frac{1}{(t-1)}$$

$$x(t) \longrightarrow -\infty \text{ as } t \longrightarrow 1$$

the solution has a *finite escape time* at $t = 1$

In general, if $f(t, x)$ is locally Lipschitz over a domain D and the solution of $\dot{x} = f(t, x)$ has a finite escape time t_e, then the solution $x(t)$ must leave every compact (closed and bounded) subset of D as $t \longrightarrow t_e$

Global Existence and Uniqueness

A function $f(t, x)$ is globally Lipschitz in x if

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\|$$

for all $x, y \in R^n$ with the same Lipschitz constant L

If $f(t, x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous for all $x \in R^n$, then $f(t, x)$ is globally Lipschitz in x if and only if the partial derivatives $\partial f_i/\partial x_j$ are globally bounded, uniformly in t

$f(x) = -x^2$ is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded

Lemma: Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$

The global Lipschitz condition is satisfied for linear systems of the form

$$\dot{x} = A(t)x + g(t)$$

but it is a restrictive condition for general nonlinear systems

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, and suppose that every solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

with $x_0 \in W$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$

Example:

$$\dot{x} = -x^3 = f(x)$$

$f(x)$ is locally Lipschitz on R, but not globally Lipschitz because $f'(x) = -3x^2$ is not globally bounded

If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive

Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in R \mid |x| \leq |a|\}$

Thus, the equation has a unique solution for all $t \geq 0$

Equilibrium Points

A point $x = x^*$ in the state space is said to be an equilibrium point of $\dot{x} = f(t, x)$ if

$$x(t_0) = x^* \;\; \Rightarrow \;\; x(t) \equiv x^*, \;\; \forall \, t \geq t_0$$

For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real solutions of the equation

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points

A linear system $\dot{x} = Ax$ can have an isolated equilibrium point at $x = 0$ (if A is nonsingular) or a continuum of equilibrium points in the null space of A (if A is singular)

It cannot have multiple isolated equilibrium points , for if x_a and x_b are two equilibrium points, then by linearity any point on the line $\alpha x_a + (1 - \alpha) x_b$ connecting x_a and x_b will be an equilibrium point

A nonlinear state equation can have multiple isolated equilibrium points .For example, the state equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 - b x_2$$

has equilibrium points at $(x_1 = n\pi, x_2 = 0)$ for $n = 0, \pm 1, \pm 2, \cdots$

Linearization

A common engineering practice in analyzing a nonlinear system is to linearize it about some nominal operating point and analyze the resulting linear model

What are the limitations of linearization?

- Since linearization is an approximation in the neighborhood of an operating point, it can only predict the “local” behavior of the nonlinear system in the vicinity of that point. It cannot predict the “nonlocal” or “global” behavior

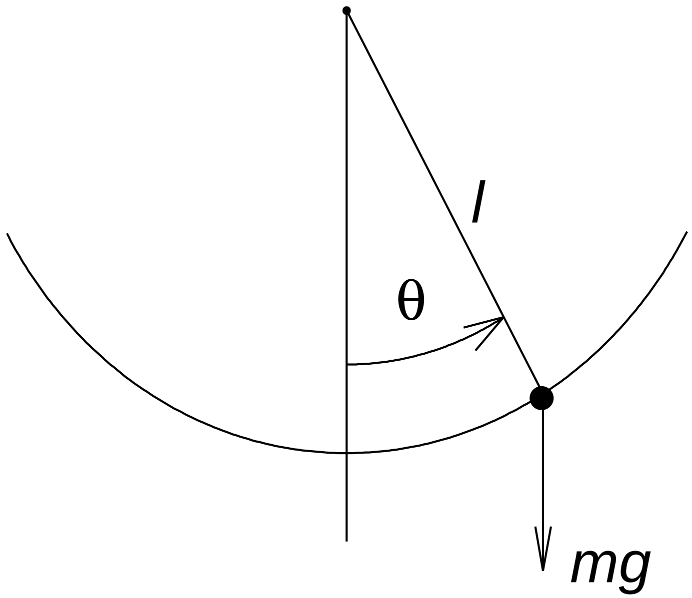
- There are “essentially nonlinear phenomena” that can take place only in the presence of nonlinearity

Nonlinear Phenomena

- Finite escape time
- Multiple isolated equilibrium points
- Limit cycles
- Subharmonic, harmonic, or almost-periodic oscillations
- Chaos
- Multiple modes of behavior

Nonlinear Systems and Control
Lecture # 2
Examples of Nonlinear Systems

Pendulum Equation



$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

$$x_1 = \theta, \quad x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

Equilibrium Points:

$$0 = x_2$$

$$0 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

$$(n\pi, 0) \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

Nontrivial equilibrium points at $(0, 0)$ and $(\pi, 0)$

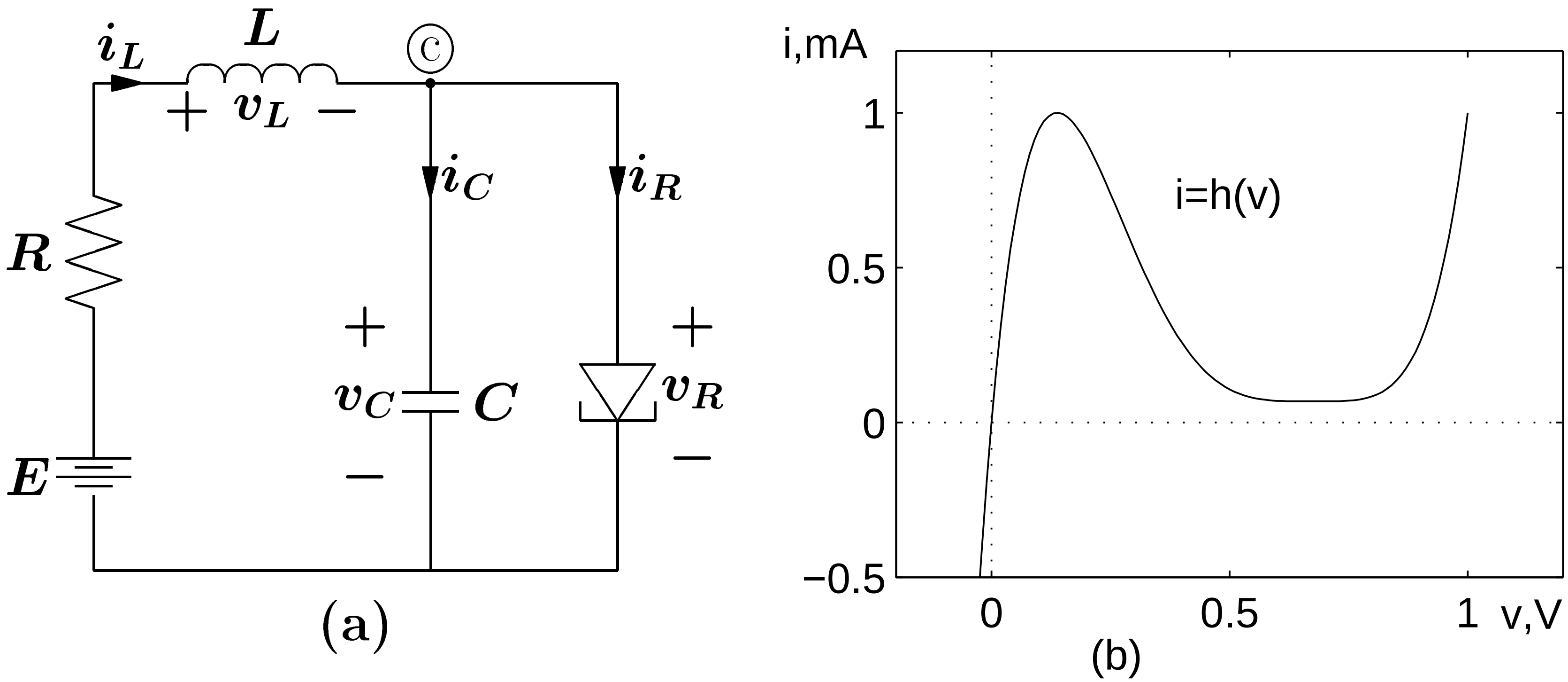
Pendulum without friction:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1\end{aligned}$$

Pendulum with torque input:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T\end{aligned}$$

Tunnel-Diode Circuit



$$i_C = C\frac{dv_C}{dt}, \qquad v_L = L\frac{di_L}{dt}$$

$$x_1 = v_C, \;\; x_2 = i_L, \quad u = E$$

$$i_C + i_R - i_L = 0 \quad \Rightarrow \quad i_C = -h(x_1) + x_2$$

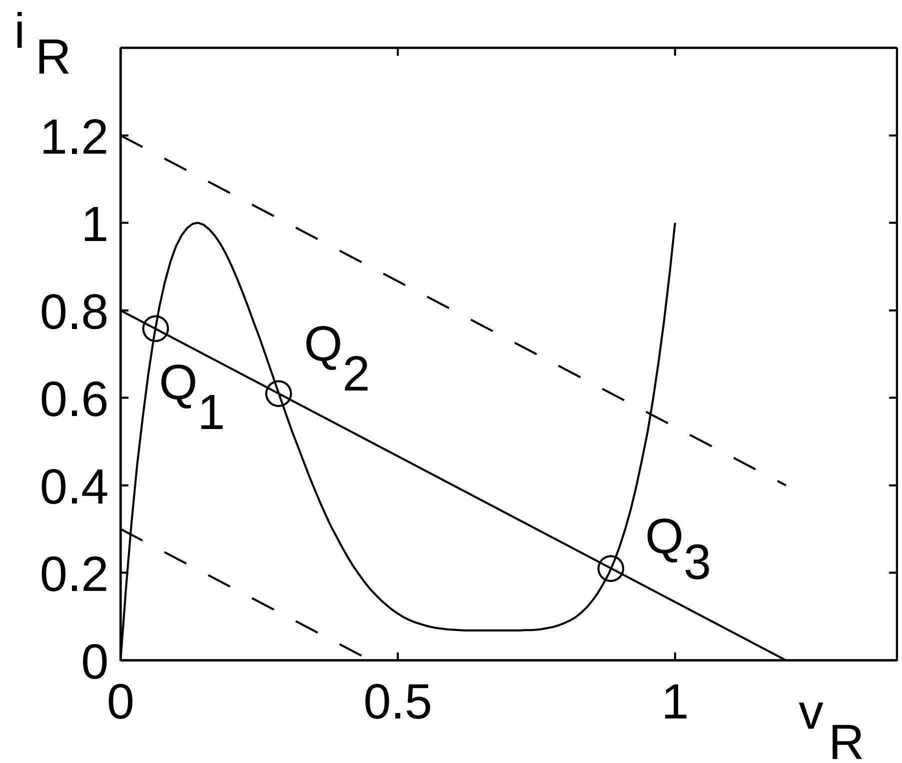
$$v_C - E + Ri_L + v_L = 0 \quad \Rightarrow \quad v_L = -x_1 - Rx_2 + u$$

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}\left[-h(x_1) + x_2\right] \\ \dot{x}_2 &= \frac{1}{L}\left[-x_1 - Rx_2 + u\right] \end{aligned}$$

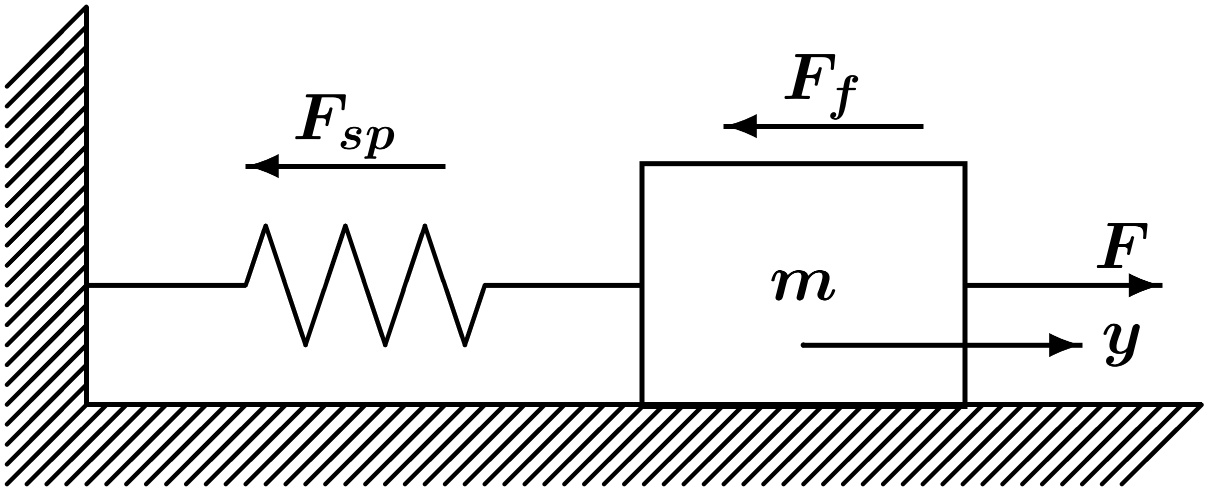
Equilibrium Points:

$$\begin{aligned} 0 &= -h(x_1) + x_2 \\ 0 &= -x_1 - Rx_2 + u \end{aligned}$$

$$h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$



Mass–Spring System



$$m\ddot{y} + F_f + F_{sp} = F$$

Sources of nonlinearity:

- Nonlinear spring restoring force $F_{sp} = g(y)$
- Static or Coulomb friction

$$F_{sp} = g(y)$$

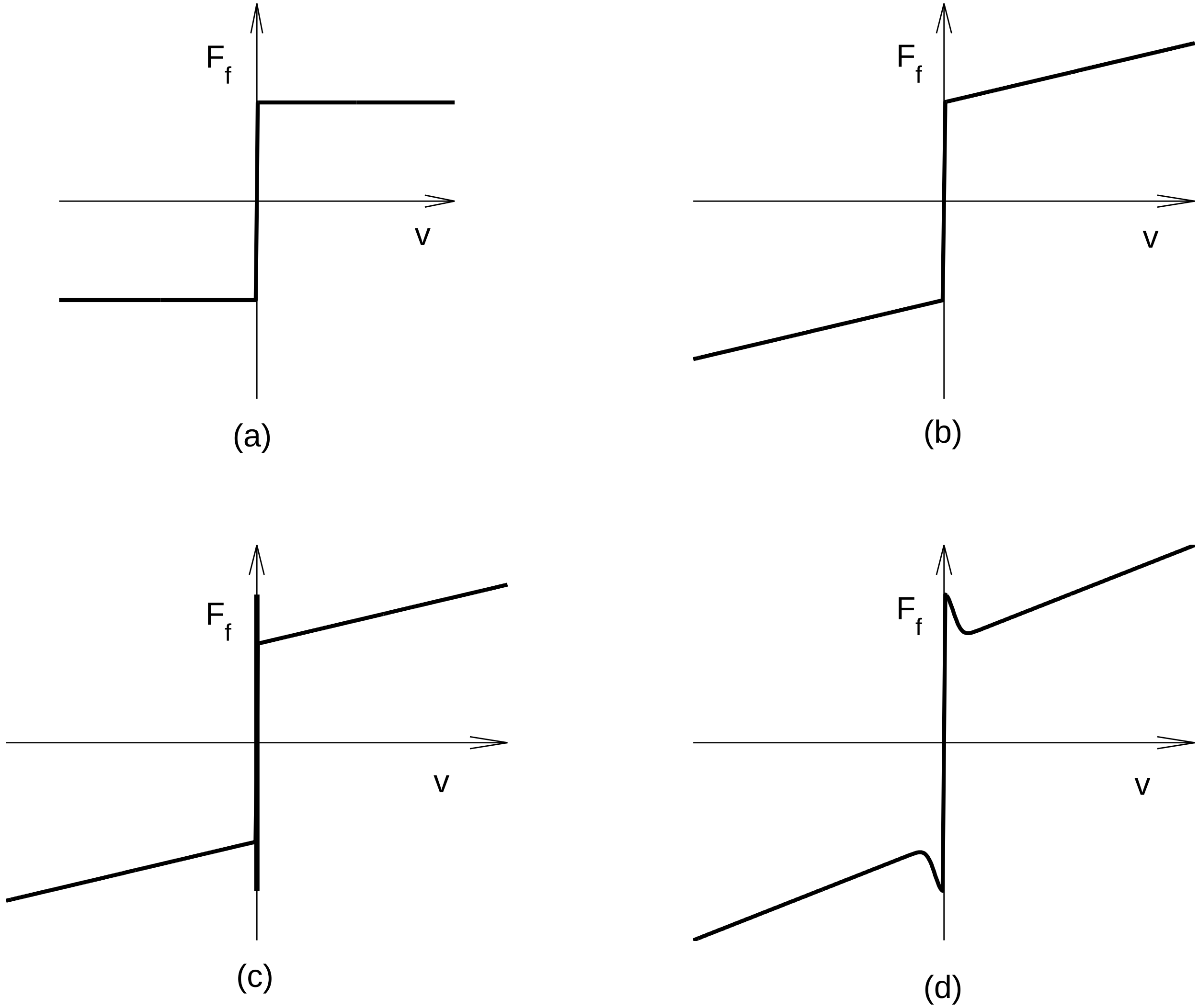
$$g(y) = k(1 - a^2y^2)y, \quad |ay| < 1 \quad \text{(softening spring)}$$

$$g(y) = k(1 + a^2y^2)y \quad \text{(hardening spring)}$$

F_f may have components due to static, Coulomb, and viscous friction

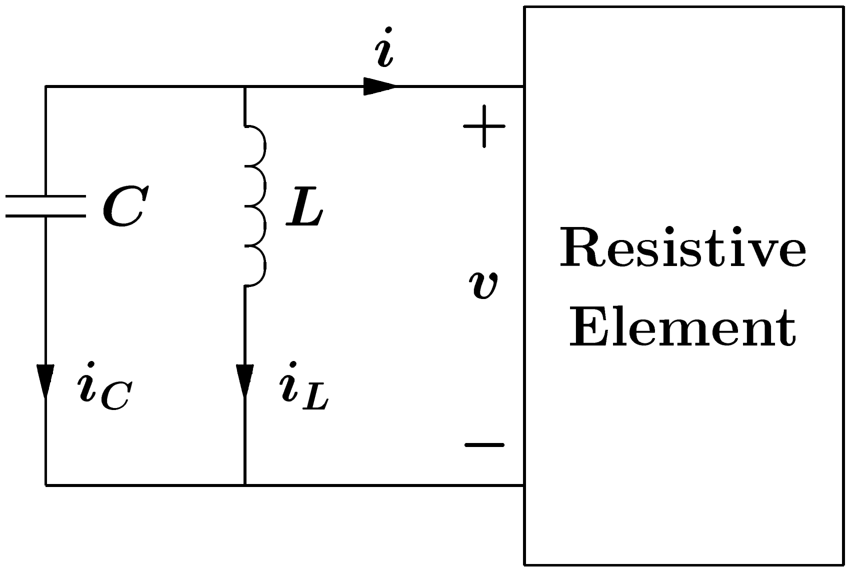
When the mass is at rest, there is a static friction force F_s that acts parallel to the surface and is limited to $\pm\mu_s mg$ ($0 < \mu_s < 1$). F_s takes whatever value, between its limits, to keep the mass at rest

Once motion has started, the resistive force F_f is modeled as a function of the sliding velocity $v = \dot{y}$

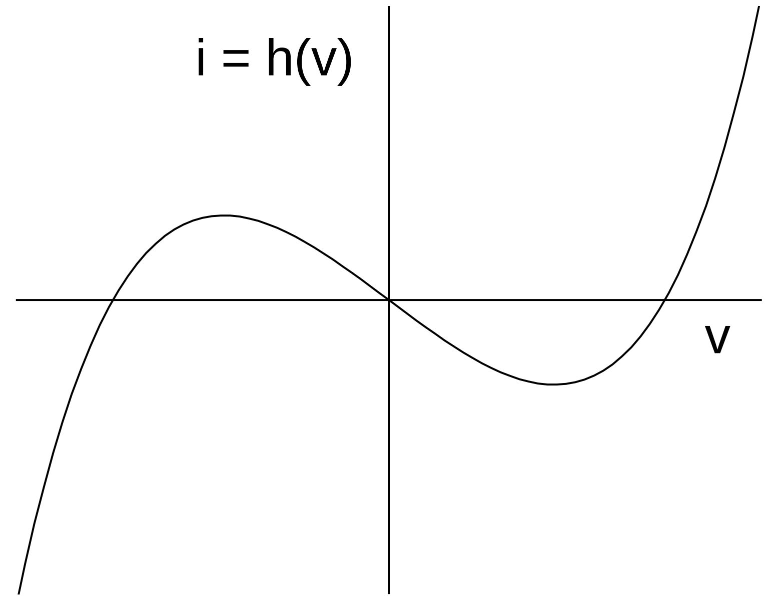


(a) Coulomb friction; (b) Coulomb plus linear viscous friction; (c) static, Coulomb, and linear viscous friction; (d) static, Coulomb, and linear viscous friction—Stribeck effect

Negative-Resistance Oscillator



(a)



(b)

$$h(0) = 0, \quad h'(0) < 0$$

$$h(v) \to \infty \text{ as } v \to \infty, \text{ and } h(v) \to -\infty \text{ as } v \to -\infty$$

$$i_C + i_L + i = 0$$

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(s)\,ds + h(v) = 0$$

Differentiating with respect to t and multiplying by L:

$$CL\frac{d^2v}{dt^2} + v + Lh'(v)\frac{dv}{dt} = 0$$

$$\tau = t/\sqrt{CL}$$

$$\frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt}, \qquad \frac{d^2v}{d\tau^2} = CL\frac{d^2v}{dt^2}$$

Denote the derivative of v with respect to τ by $\dot{v}$

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0, \quad \varepsilon = \sqrt{L/C}$$

Special case: Van der Pol equation

$$h(v) = -v + \tfrac{1}{3}v^3$$

$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0$$

State model: $x_1 = v, \quad x_2 = \dot{v}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2 \end{aligned}$$

Another State Model: $z_1 = i_L, \quad z_2 = v_C$

$$\begin{aligned} \dot{z}_1 &= \frac{1}{\varepsilon} z_2 \\ \dot{z}_2 &= -\varepsilon[z_1 + h(z_2)] \end{aligned}$$

Change of variables: $z = T(x)$

$$\begin{aligned} x_1 &= v = z_2 \\ x_2 &= \frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt} = \sqrt{\frac{L}{C}}[-i_L - h(v_C)] \\ &= \varepsilon[-z_1 - h(z_2)] \end{aligned}$$

$$T(x) = \begin{bmatrix} -h(x_1) - \frac{1}{\varepsilon} x_2 \\ x_1 \end{bmatrix}, \; T^{-1}(z) = \begin{bmatrix} z_2 \\ -\varepsilon z_1 - \varepsilon h(z_2) \end{bmatrix}$$

Adaptive Control

$$\boldsymbol{Plant:} \qquad \dot{y}_p = a_p y_p + k_p u$$

$$\boldsymbol{ReferenceModel:} \qquad \dot{y}_m = a_m y_m + k_m r$$

$$u(t) = \theta_1^* r(t) + \theta_2^* y_p(t)$$

$$\theta_1^* = \frac{k_m}{k_p} \quad \text{and} \quad \theta_2^* = \frac{a_m - a_p}{k_p}$$

When a_p and k_p are unknown, we may use

$$u(t) = \theta_1(t) r(t) + \theta_2(t) y_p(t)$$

where $\theta_1(t)$ and $\theta_2(t)$ are adjusted on-line

Adaptive Law (gradient algorithm):

$$\begin{aligned}\dot{\theta}_1 &= -\gamma(y_p - y_m)r \\ \dot{\theta}_2 &= -\gamma(y_p - y_m)y_p, \quad \gamma > 0\end{aligned}$$

State Variables: $e_o = y_p - y_m,\ \phi_1 = \theta_1 - \theta_1^*,\ \phi_2 = \theta_2 - \theta_2^*$

$$\dot{y}_m = a_p y_m + k_p(\theta_1^* r + \theta_2^* y_m)$$

$$\dot{y}_p = a_p y_p + k_p(\theta_1 r + \theta_2 y_p)$$

$$\begin{aligned}\dot{e}_o &= a_p e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 y_p - \theta_2^* y_m) \\ &= \cdots\cdots + k_p[\theta_2^* y_p - \theta_2^* y_p] \\ &= (a_p + k_p\theta_2^*)e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 - \theta_2^*)y_p\end{aligned}$$

Closed-Loop System:

$$
\begin{aligned}
\dot{e}_o &= a_m e_o + k_p \phi_1 r(t) + k_p \phi_2 [e_o + y_m(t)] \\
\dot{\phi}_1 &= -\gamma e_o r(t) \\
\dot{\phi}_2 &= -\gamma e_o [e_o + y_m(t)]
\end{aligned}
$$

Nonlinear Systems and Control
Lecture # 3
Second-Order Systems

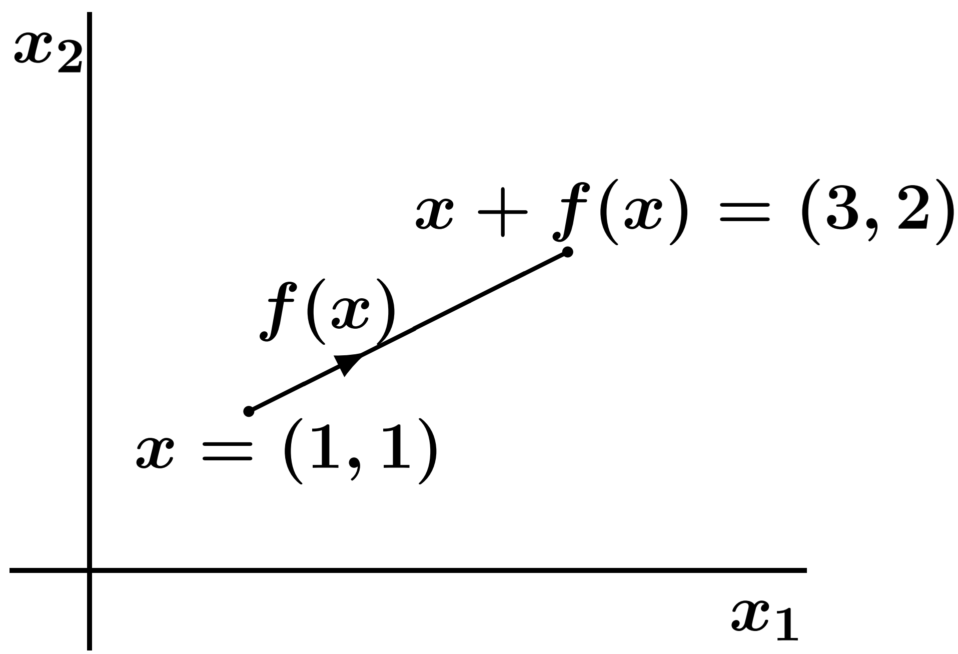
$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) = f_1(x) \\
\dot{x}_2 &= f_2(x_1, x_2) = f_2(x)
\end{aligned}
$$

Let $x(t) = (x_1(t), x_2(t))$ be a solution that starts at initial state $x_0 = (x_{10}, x_{20})$. The locus in the x_1–x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit*
The x_1–x_2 plane is called the *state plane* or *phase plane*
The family of all trajectories is called the *phase portrait*
The *vector field* $f(x) = (f_1(x), f_2(x))$ is tangent to the trajectory at point x because

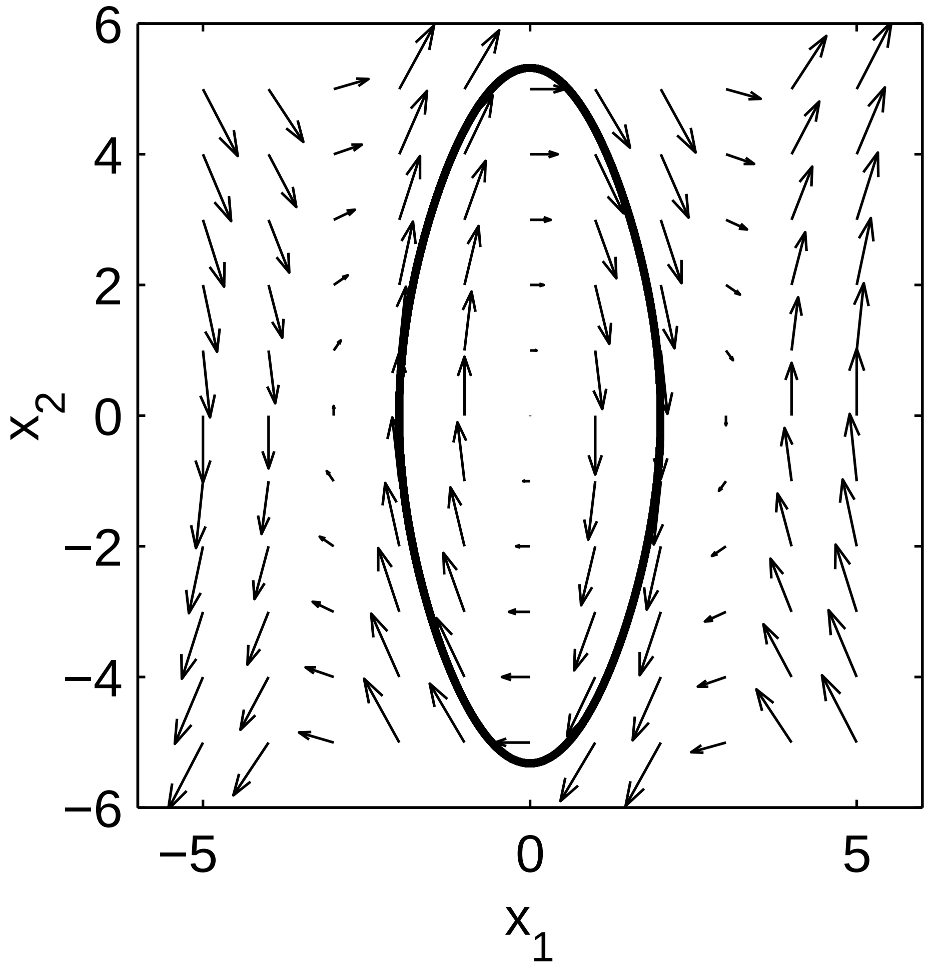
$$
\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}
$$

Vector Field diagram

Represent $f(x)$ as a vector based at x; that is, assign to x the directed line segment from x to $x + f(x)$



Repeat at every point in a grid covering the plane



$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -10 \sin x_1$$

Numerical Construction of the Phase Portrait:

- Select a bounding box in the state plane
- Select an initial point x_0 and calculate the trajectory through it by solving

$$\dot{x} = f(x), \quad x(0) = x_0$$

in forward time (with positive t) and in reverse time (with negative t)

$$\dot{x} = -f(x), \quad x(0) = x_0$$

- Repeat the process interactively

Use Simulink or pplane

Qualitative Behavior of Linear Systems

$$\dot{x} = Ax, \qquad A \text{ is a } 2 \times 2 \text{ real matrix}$$

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

$$J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$x(t) = Mz(t)$$

$$\dot{z} = J_r z(t)$$

Case 1. Both eigenvalues are real: $\lambda_1 \neq \lambda_2 \neq 0$

$$M = [v_1, v_2]$$

v_1 & v_2 are the real eigenvectors associated with λ_1 & λ_2

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2$$

$$z_1(t) = z_{10} e^{\lambda_1 t}, \qquad z_2(t) = z_{20} e^{\lambda_2 t}$$

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \quad c = z_{20}/(z_{10})^{\lambda_2/\lambda_1}$$

The shape of the phase portrait depends on the signs of λ_1 and λ_2

$$\lambda_2 < \lambda_1 < 0$$

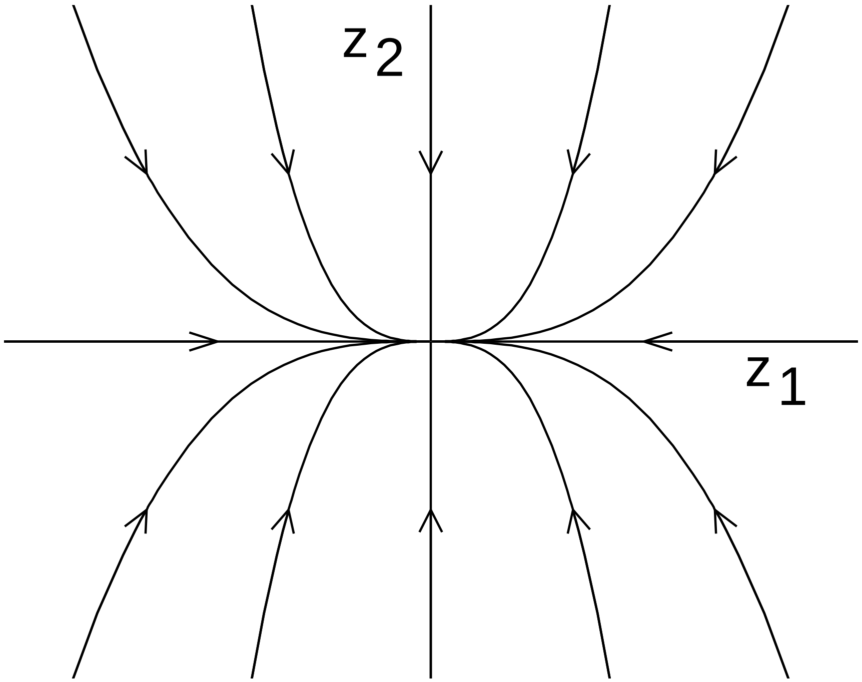
$e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero as $t \to \infty$

$e^{\lambda_2 t}$ tends to zero faster than $e^{\lambda_1 t}$

Call λ_2 the fast eigenvalue (v_2 the fast eigenvector) and λ_1 the slow eigenvalue (v_1 the slow eigenvector)

The trajectory tends to the origin along the curve $z_2 = c z_1^{\lambda_2/\lambda_1}$ with $\lambda_2/\lambda_1 > 1$

$$\frac{dz_2}{dz_1} = c\frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]}$$

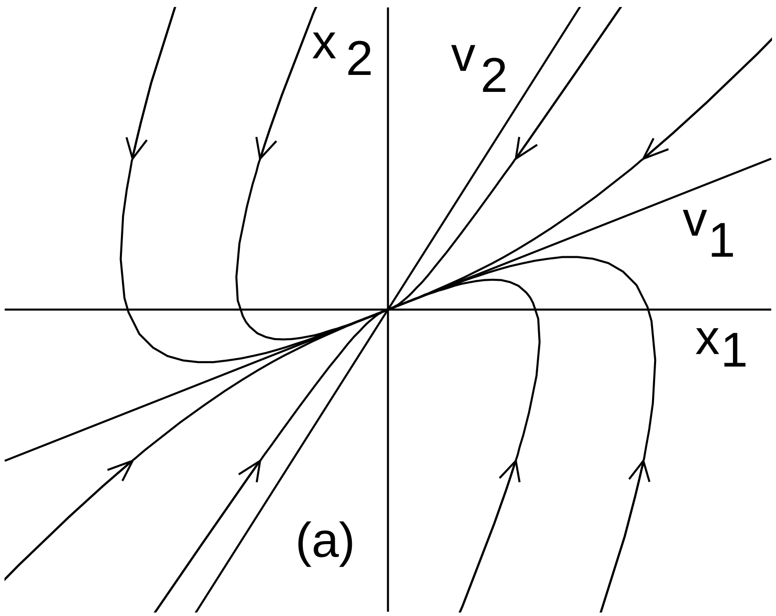


Stable Node

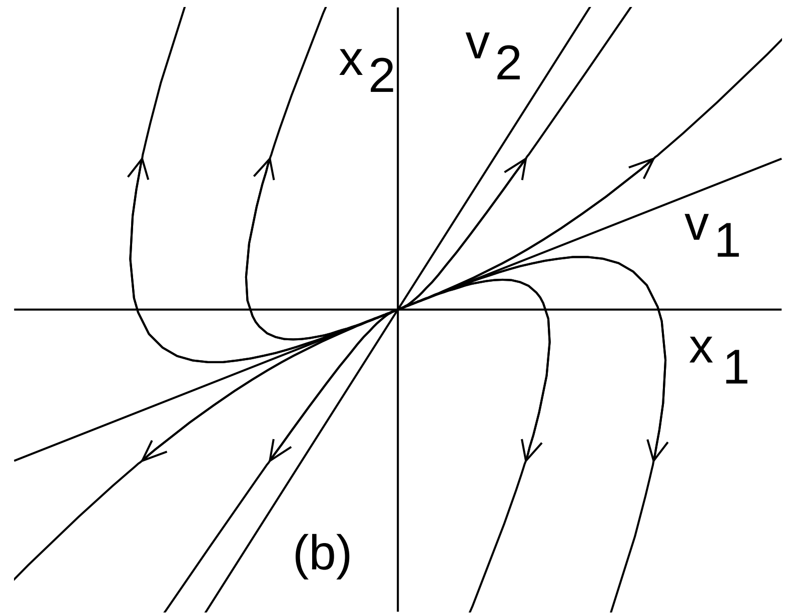
$$\lambda_2 > \lambda_1 > 0$$

Reverse arrowheads

Reverse arrowheads $\Longrightarrow$ Unstable Node



Stable Node



Unstable Node

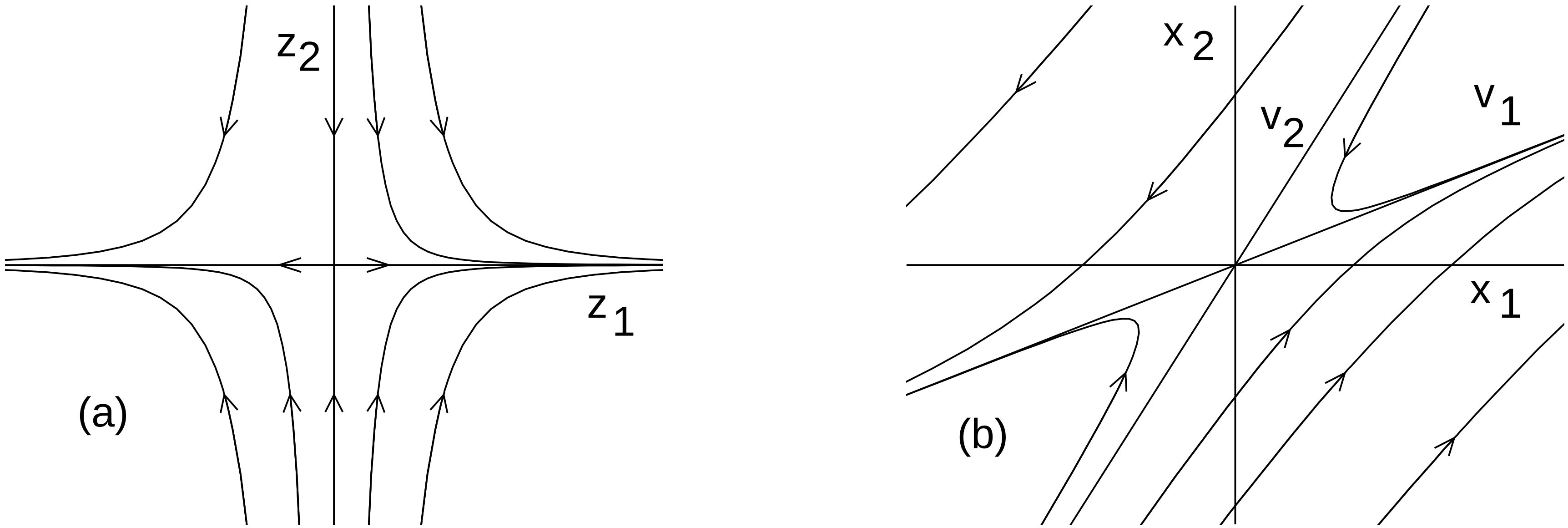
$$\lambda_2 < 0 < \lambda_1$$

$e^{\lambda_1 t} \longrightarrow \infty$, while $e^{\lambda_2 t} \longrightarrow 0$ as $t \longrightarrow \infty$

Call λ_2 the stable eigenvalue (v_2 the stable eigenvector) and λ_1 the unstable eigenvalue (v_1 the unstable eigenvector)

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \quad \lambda_2/\lambda_1 < 0$$

Saddle



Phase Portrait of a Saddle Point

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$$

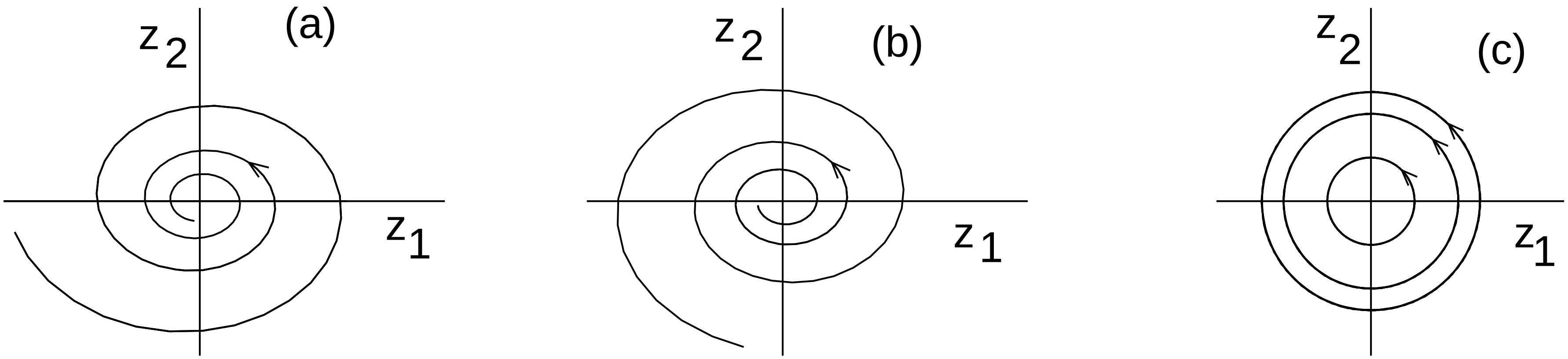
$$r = \sqrt{z_1^2 + z_2^2}, \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

$$r(t) = r_0 e^{\alpha t} \quad \text{and} \quad \theta(t) = \theta_0 + \beta t$$

$$\alpha < 0 \;\Rightarrow\; r(t) \to 0 \text{ as } t \to \infty$$

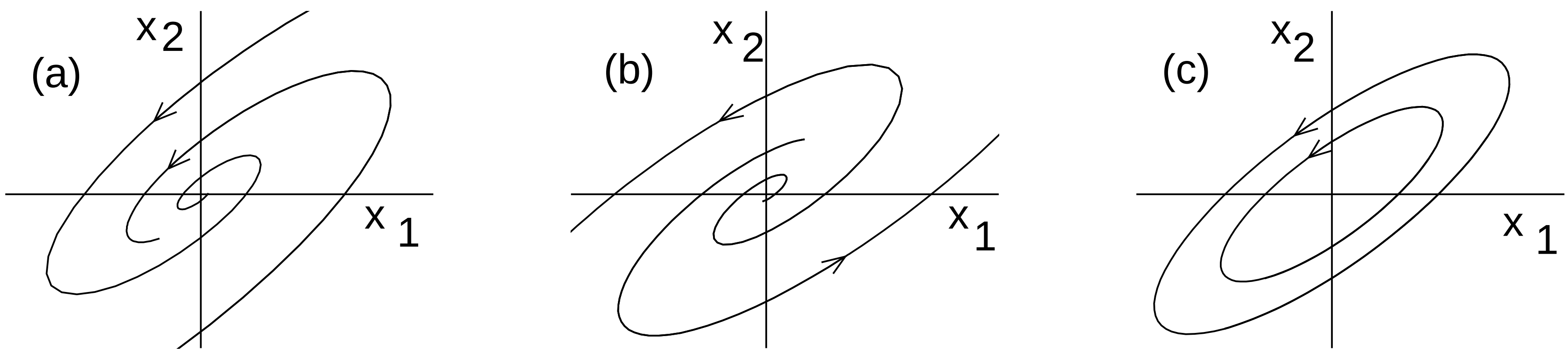
$$\alpha > 0 \;\Rightarrow\; r(t) \to \infty \text{ as } t \to \infty$$

$$\alpha = 0 \;\Rightarrow\; r(t) \equiv r_0 \;\forall\, t$$



$\alpha < 0$ $\alpha > 0$ $\alpha = 0$

Stable Focus | Unstable Focus | Center



Effect of Perturbations

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} + \boldsymbol{\delta A} \quad (\boldsymbol{\delta A} \text{ arbitrarily small})$$

The eigenvalues of a matrix depend continuously on its parameters

A node (with distinct eigenvalues), a saddle or a focus is structurally stable because the qualitative behavior remains the same under arbitrarily small perturbations in $\boldsymbol{A}$

A stable node with multiple eigenvalues could become a stable node or a stable focus under arbitrarily small perturbations in $\boldsymbol{A}$

A center is not structurally stable

$$\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$$

$$\text{Eigenvalues} = \mu \pm j$$

$$\mu < 0 \quad \Rightarrow \quad \text{Stable Focus}$$

$$\mu > 0 \quad \Rightarrow \quad \text{Unstable Focus}$$

Nonlinear Systems and Control
Lecture # 4
Qualitative Behavior Near Equilibrium Points
&
Multiple Equilibria

The qualitative behavior of a nonlinear system near an equilibrium point can take one of the patterns we have seen with linear systems. Correspondingly the equilibrium points are classified as stable node, unstable node, saddle, stable focus, unstable focus, or center

Can we determine the type of the equilibrium point of a nonlinear system by linearization?

Let $p = (p_1, p_2)$ be an equilibrium point of the system

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are continuously differentiable
Expand f_1 and f_2 in Taylor series about (p_1, p_2)

$$\begin{aligned} \dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.} \\ \dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.} \end{aligned}$$

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$a_{21} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{22} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$f_1(p_1, p_2) = f_2(p_1, p_2) = 0$$

$$y_1 = x_1 - p_1 \qquad y_2 = x_2 - p_2$$

$$\begin{aligned} \dot{y}_1 &= \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + \text{H.O.T.} \\ \dot{y}_2 &= \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + \text{H.O.T.} \end{aligned}$$

$$\dot{y} \approx Ay$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Bigg|_{x=p} = \frac{\partial f}{\partial x}\Big|_{x=p}$$

Eigenvalues of A	Type of equilibrium point of the nonlinear system
$\lambda_2 < \lambda_1 < 0$	Stable Node
$\lambda_2 > \lambda_1 > 0$	Unstable Node
$\lambda_2 < 0 < \lambda_1$	Saddle
$\alpha \pm j\beta,\ \alpha < 0$	Stable Focus
$\alpha \pm j\beta,\ \alpha > 0$	Unstable Focus
$\pm j\beta$	Linearization Fails

Example

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 - \mu x_2(x_1^2 + x_2^2)$$

$x = 0$ is an equilibrium point

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\mu(3x_1^2 + x_2^2) & -(1 + 2\mu x_1 x_2) \\ (1 - 2\mu x_1 x_2) & -\mu(x_1^2 + 3x_2^2) \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$x_1 = r\cos\theta \text{ and } x_2 = r\sin\theta \;\Rightarrow\; \dot{r} = -\mu r^3 \text{ and } \dot{\theta} = 1$$

Stable focus when $\mu > 0$ and Unstable focus when $\mu < 0$

For a saddle point, we can use linearization to generate the stable and unstable trajectories

Let the eigenvalues of the linearization be $\boldsymbol{\lambda_1 > 0 > \lambda_2}$ and the corresponding eigenvectors be $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$

The stable and unstable trajectories will be tangent to the stable and unstable eigenvectors, respectively, as they approach the equilibrium point $\boldsymbol{p}$

For the unstable trajectories use $\boldsymbol{x_0 = p \pm \alpha v_1}$

For the stable trajectories use $\boldsymbol{x_0 = p \pm \alpha v_2}$

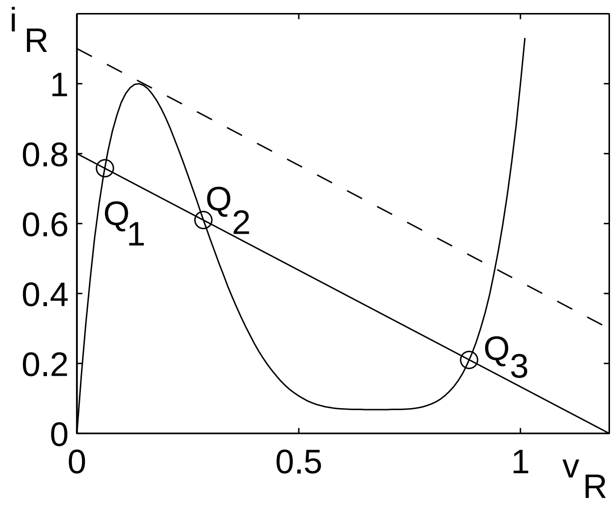
$\boldsymbol{\alpha}$ is a small positive number

Multiple Equilibria

Example: Tunnel-diode circuit

$$\begin{aligned} \dot{x}_1 &= 0.5[-h(x_1) + x_2] \\ \dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2) \end{aligned}$$

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$



$$\begin{aligned} Q_1 &= (0.063, 0.758) \\ Q_2 &= (0.285, 0.61) \\ Q_3 &= (0.884, 0.21) \end{aligned}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

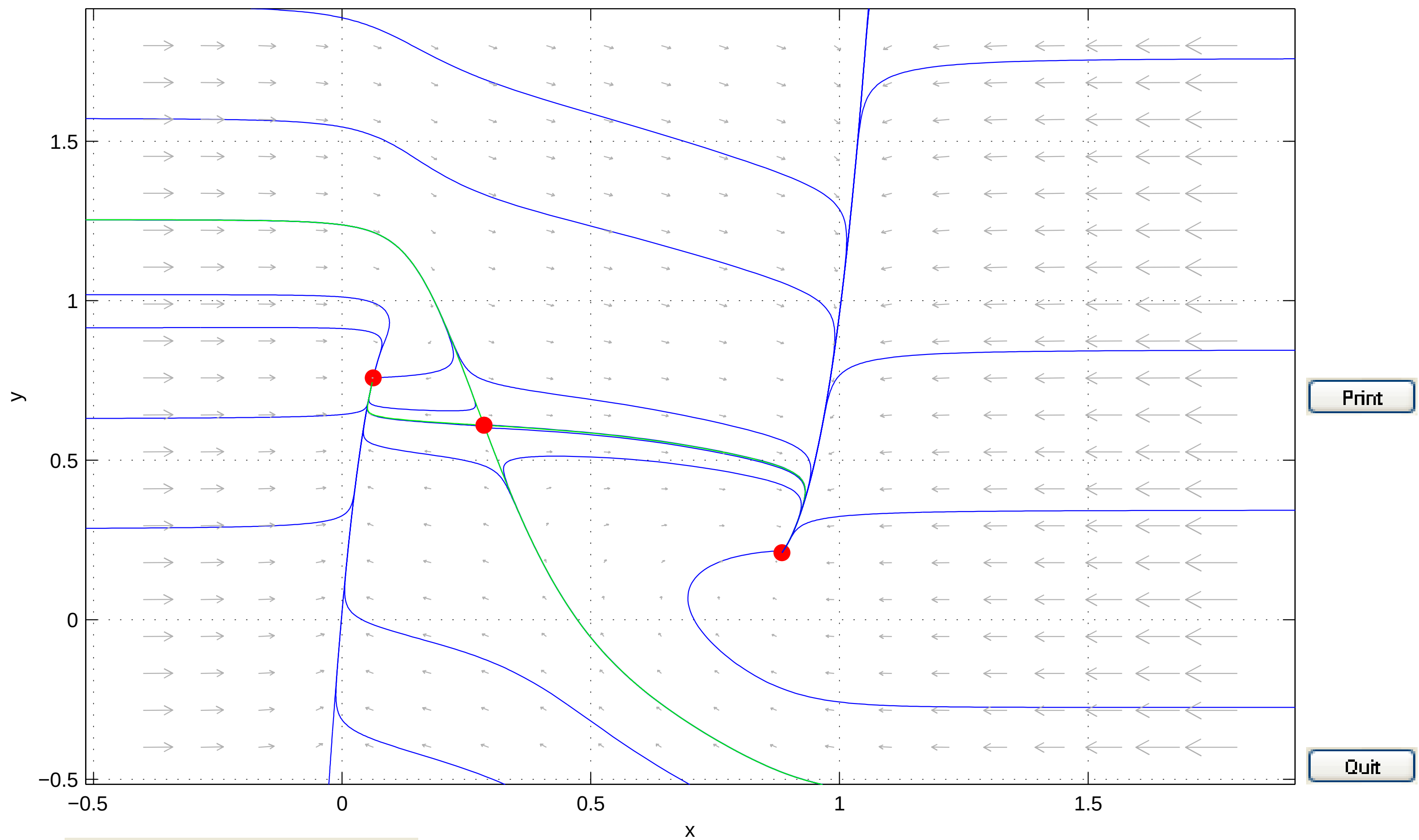
$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues}: -3.57, \ -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues}: 1.77, \ -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues}: -1.33, \ -0.4$$

Q_1 is a stable node; Q_2 is a saddle; Q_3 is a stable node

$x' = 0.5\,(-17.76\,x + 103.79\,x^2 - 229.62\,x^3 + 226.31\,x^4 - 83.72\,x^5 + y)$
$y' = 0.2\,(-x - 1.5\,y + 1.2)$

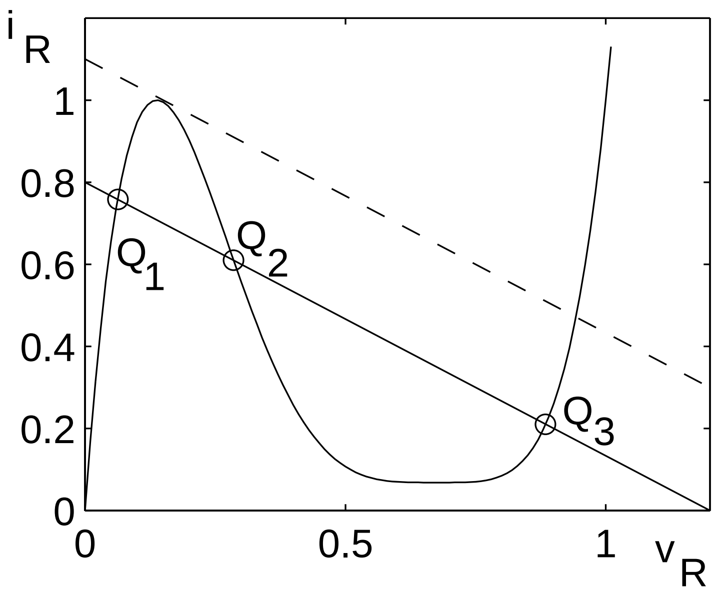


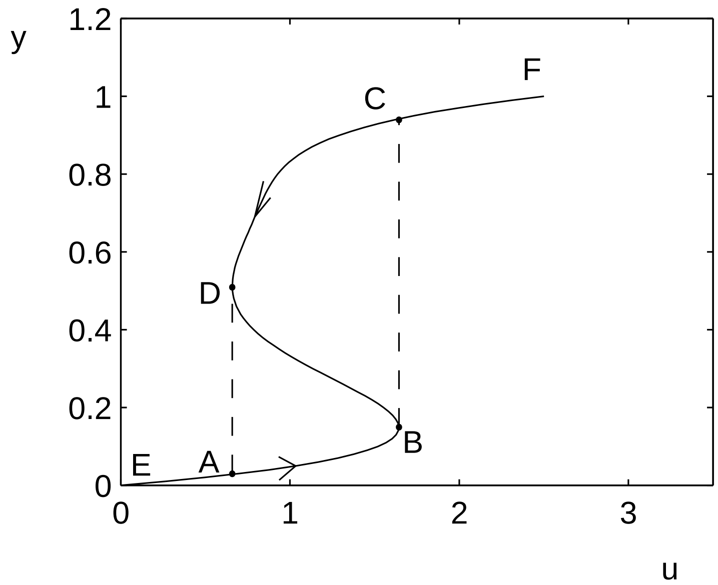
Cursor position: (1.02, -0.908)

The second unstable trajectory --> a possible eq. pt. near (0.063, 0.76).
Ready.
The forward orbit from (1.7, 2.2) --> a possible eq. pt. near (0.88, 0.21).
The backward orbit from (1.7, 2.2) left the computation window.
Ready.

Hysteresis characteristics of the tunnel-diode circuit

$$u = E, \quad y = v_R$$





Nonlinear Systems and Control
Lecture # 5
Limit Cycles

Oscillation: A system oscillates when it has a nontrivial periodic solution

$$x(t+T) = x(t), \ \ \forall\, t \geq 0$$

Linear (Harmonic) Oscillator:

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$$

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \qquad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \qquad \theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right]$$

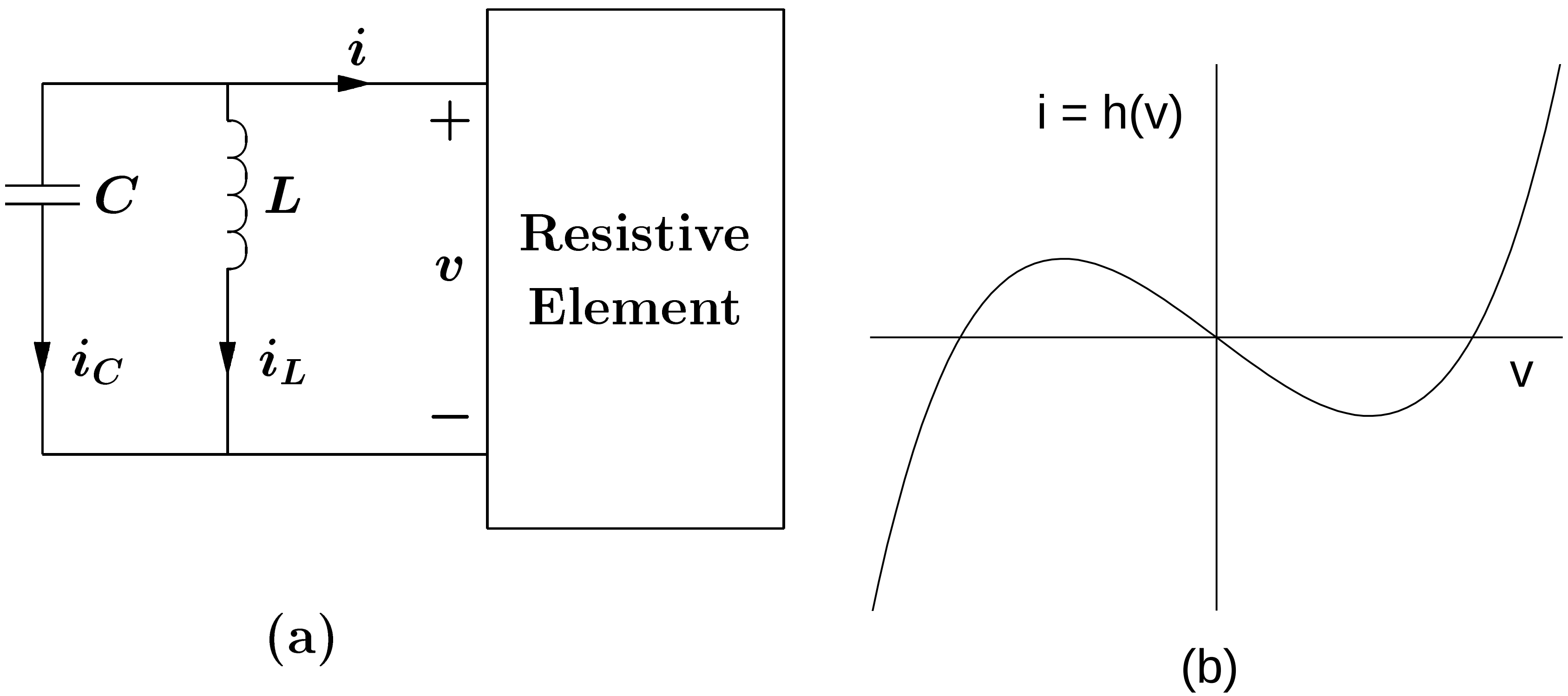
The linear oscillation is not practical because

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation)
- The amplitude of oscillation depends on the initial conditions

The same problems exist with oscillation of nonlinear systems due to a center equilibrium point (e.g., pendulum without friction)

Limit Cycles:

Example: Negative Resistance Oscillator



$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2
\end{aligned}
$$

There is a unique equilibrium point at the origin

$$
A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}
$$

$$
\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0
$$

$h'(0) < 0 \;\Rightarrow\;$ Unstable Focus or Unstable Node

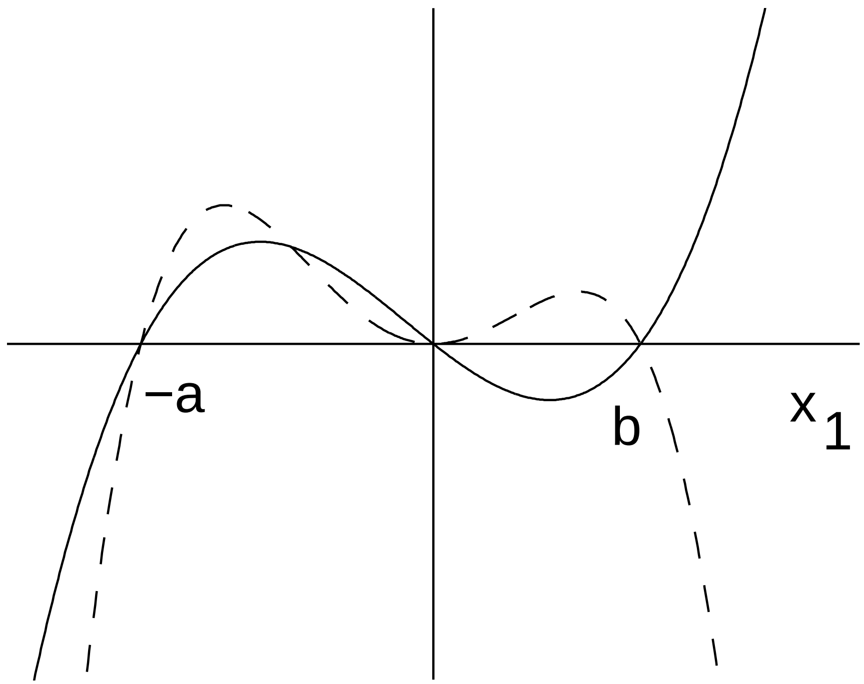
Energy Analysis:

$$E = \tfrac{1}{2}Cv_C^2 + \tfrac{1}{2}Li_L^2$$

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon}x_2$$

$$E = \tfrac{1}{2}C\{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

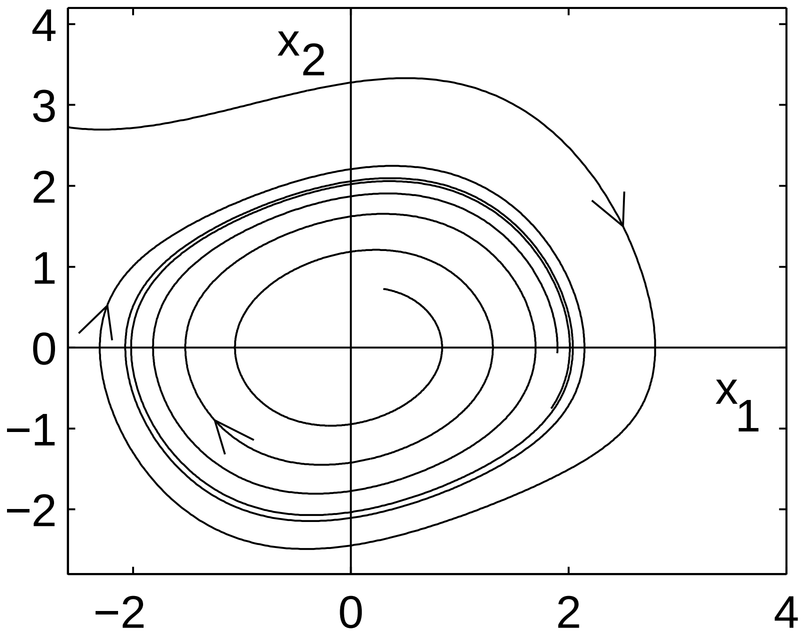
$$\begin{aligned}
\dot{E} &= C\{x_1\dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\
&= C\{x_1x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)x_2 - x_1 - \varepsilon h'(x_1)x_2]\} \\
&= C[x_1x_2 - \varepsilon x_1 h(x_1) - x_1x_2] \\
&= -\varepsilon C x_1 h(x_1)
\end{aligned}$$



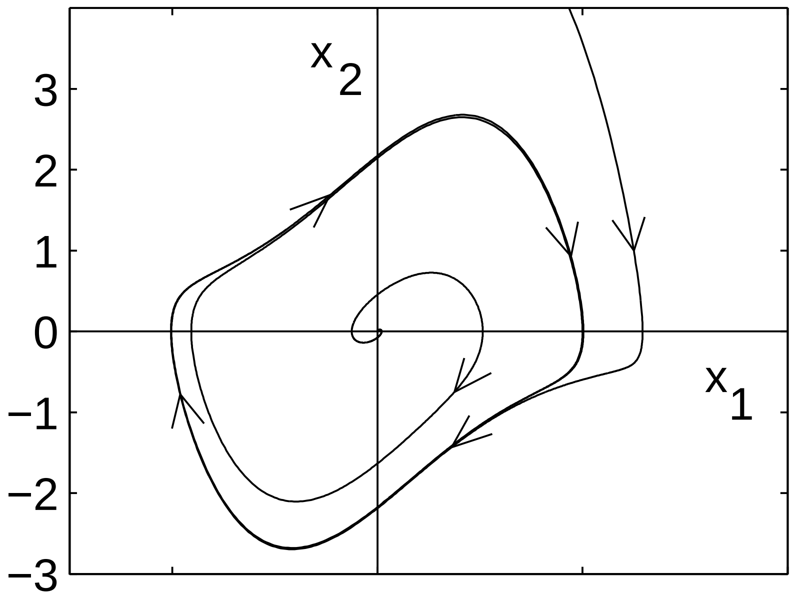
$$\dot{\boldsymbol{E}} = -\varepsilon C x_1 h(x_1)$$

Example: Van der Pol Oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2$$



(a)

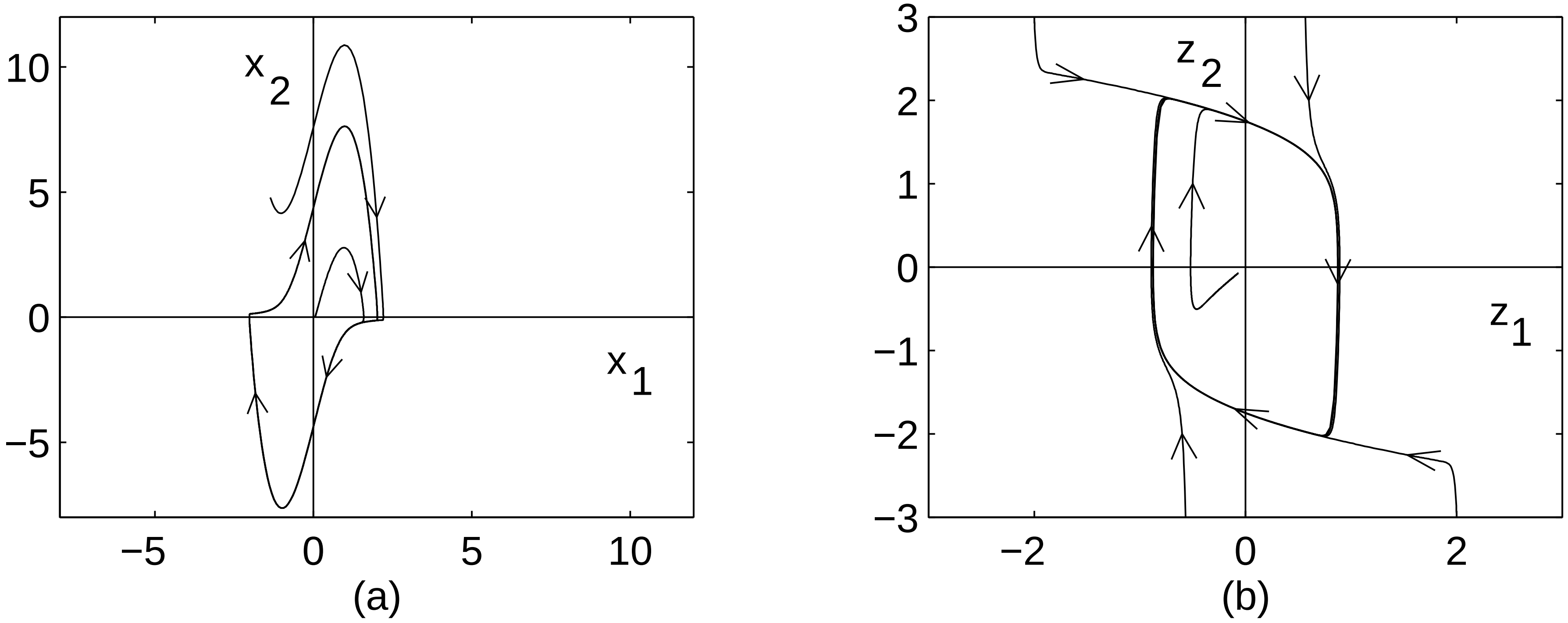


(b)

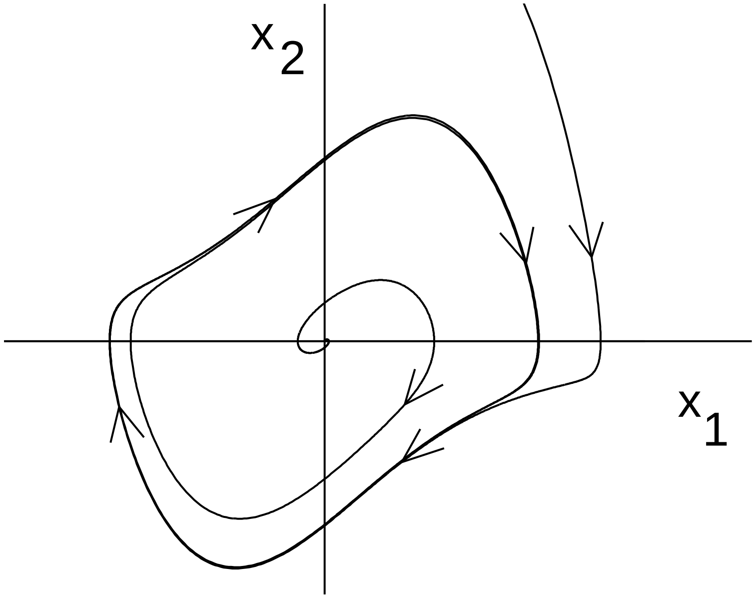
$\varepsilon = 0.2$ $\varepsilon = 1$

$$\dot{z}_1 = \frac{1}{\varepsilon} z_2$$

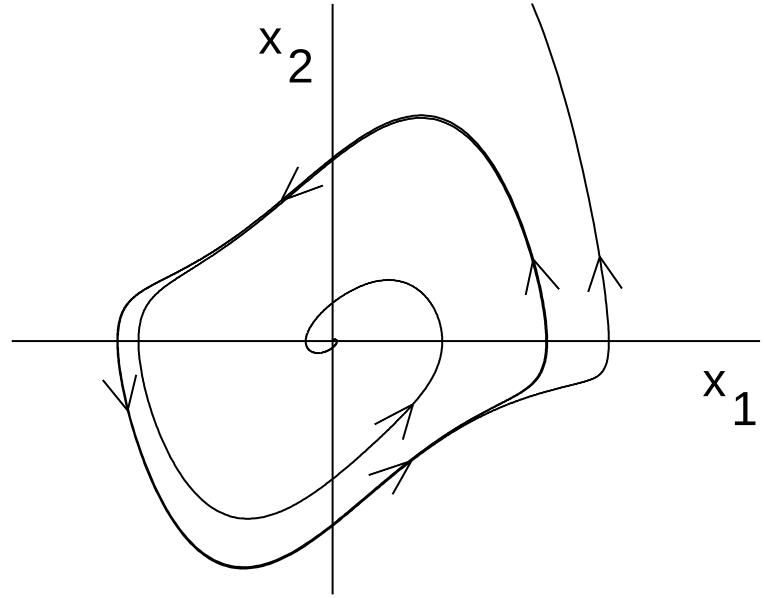
$$\dot{z}_2 = -\varepsilon(z_1 - z_2 + \tfrac{1}{3} z_2^3)$$



$\varepsilon = 5$



(a)

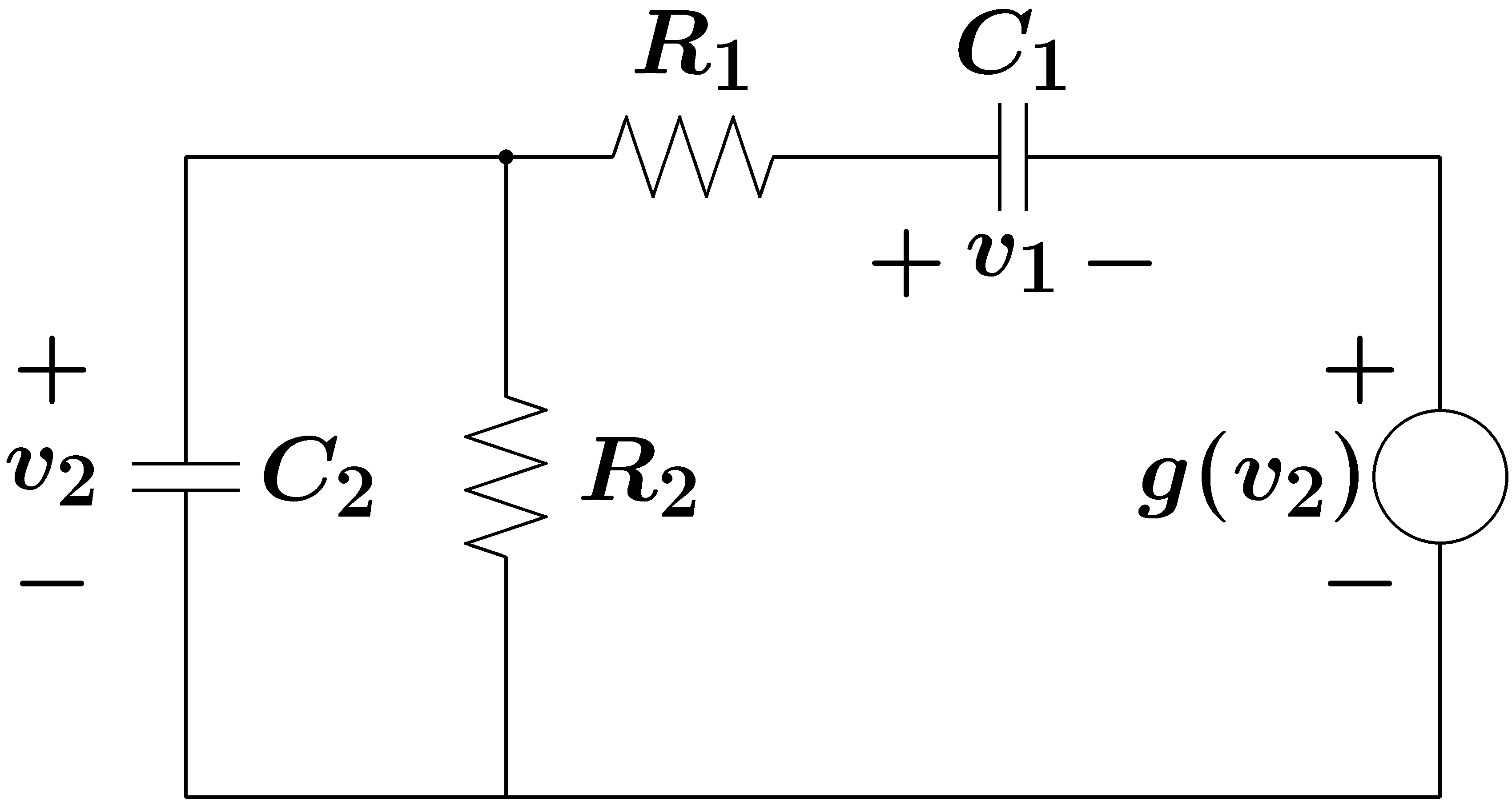


(b)

Stable Limit Cycle

Unstable Limit Cycle

Example: Wien-Bridge Oscillator



Equivalent Circuit

State variables $x_1 = v_1$ and $x_2 = v_2$

$$\dot{x}_1 = \frac{1}{C_1 R_1}[-x_1 + x_2 - g(x_2)]$$

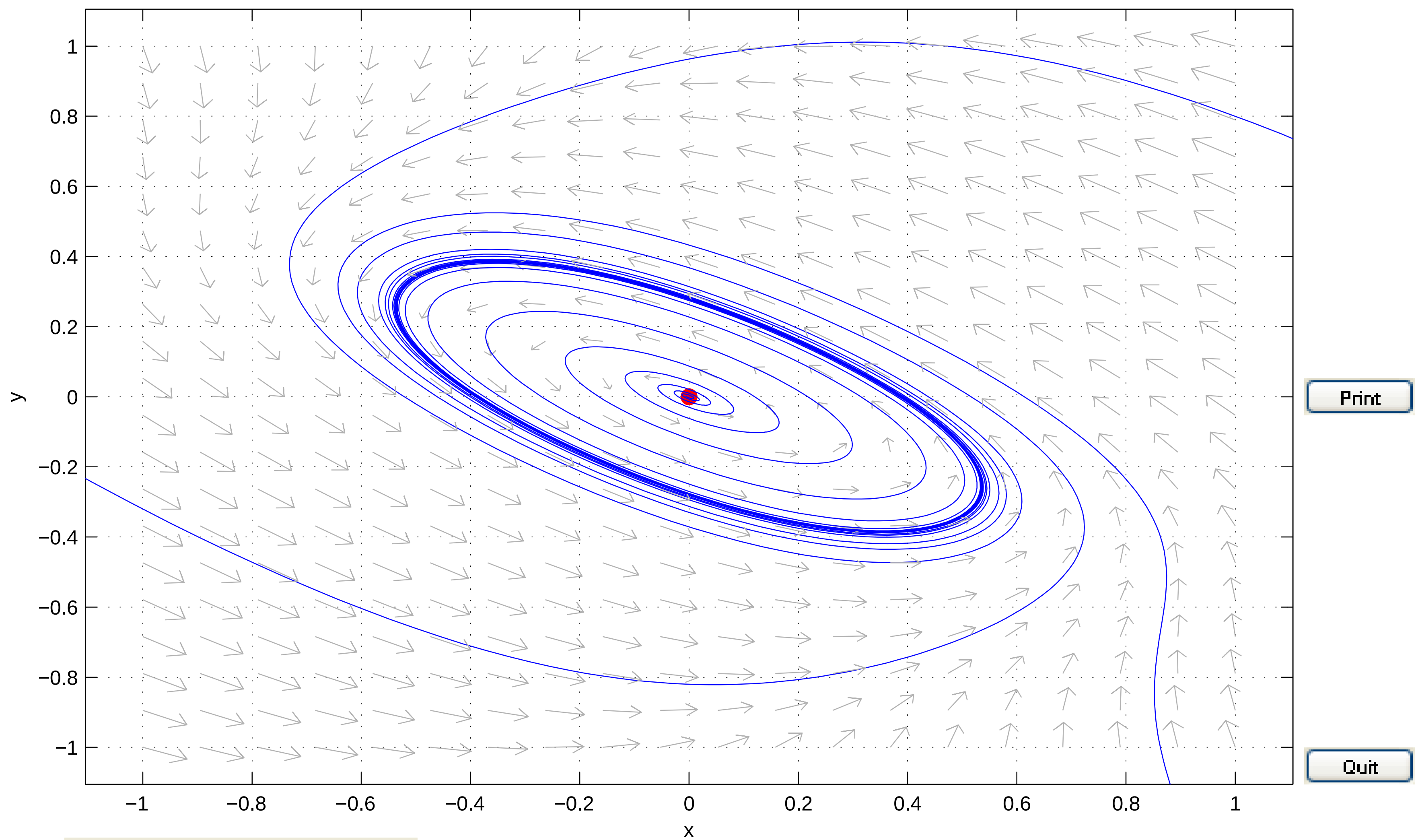
$$\dot{x}_2 = -\frac{1}{C_2 R_1}[-x_1 + x_2 - g(x_2)] - \frac{1}{C_2 R_2} x_2$$

There is a unique equilibrium point at $x = 0$

Numerical data: $C_1 = C_2 = R_1 = R_2 = 1$

$$g(v) = 3.234v - 2.195v^3 + 0.666v^5$$

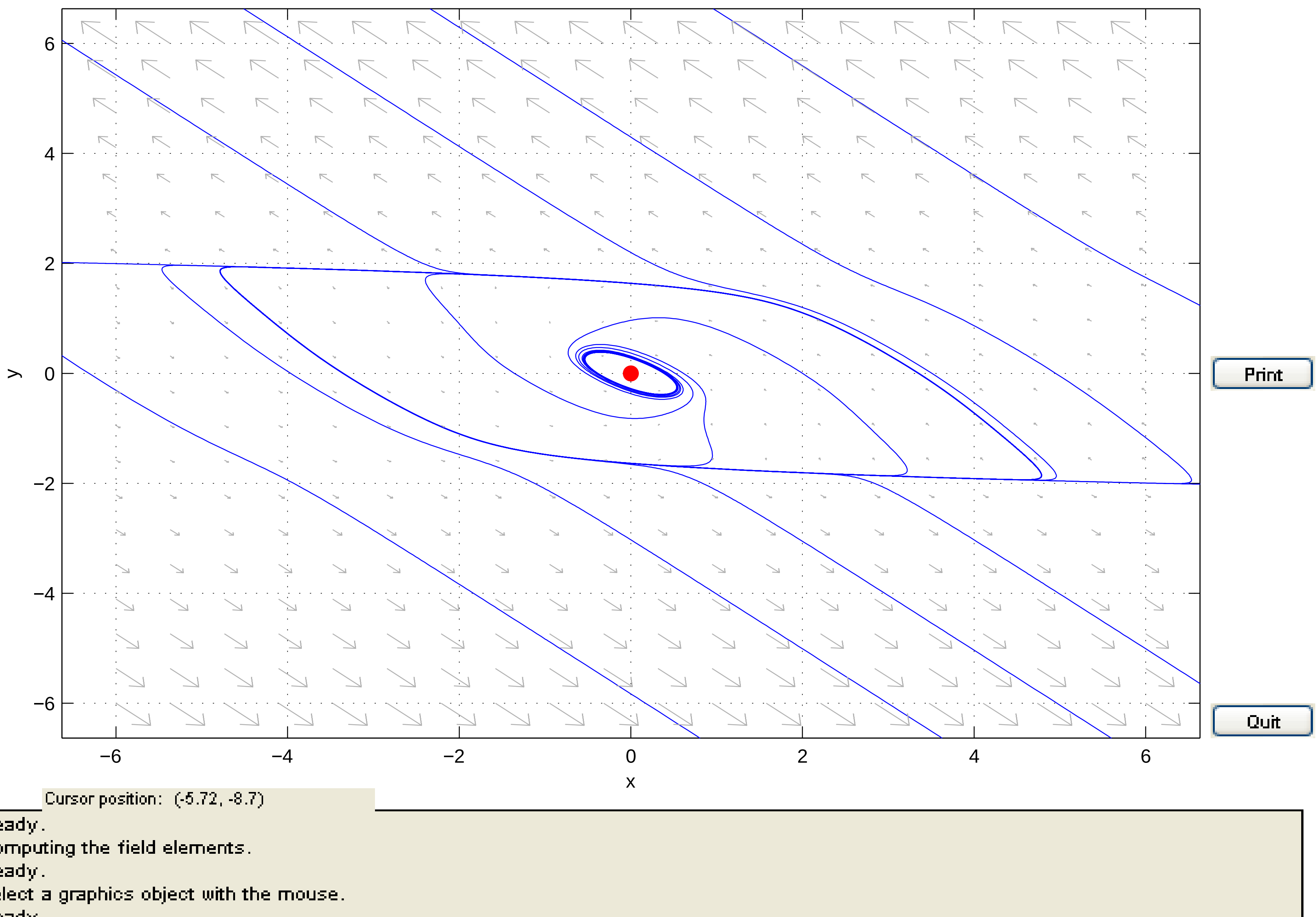
$x' = -x + y - (3.234\,y - 2.195\,y^3 + 0.666\,y^5)$
$y' = -(-x + y - (3.234\,y - 2.195\,y^3 + 0.666\,y^5)) - y$



Cursor position: (-0.663, -1.72)

Computing the field elements.
Ready.
The forward orbit from (0.13, -0.1) --> a nearly closed orbit.
The backward orbit from (0.13, -0.1) --> a possible eq. pt. near (0, 0).
Ready.

$$x' = -x + y - (3.234\,y - 2.195\,y^3 + 0.666\,y^5)$$
$$y' = -(-x + y - (3.234\,y - 2.195\,y^3 + 0.666\,y^5)) - y$$



Nonlinear Systems and Control
Lecture # 6
Bifurcation

Bifurcation is a change in the equilibrium points or periodic orbits, or in their stability properties, as a parameter is varied

Example

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

Find the equilibrium points and their types for different values of μ

For $\mu > 0$ there are two equilibrium points at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$

Linearization at $(\sqrt{\mu}, 0)$:

$$\begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

$(\sqrt{\mu}, 0)$ is a stable node

Linearization at $(-\sqrt{\mu}, 0)$:

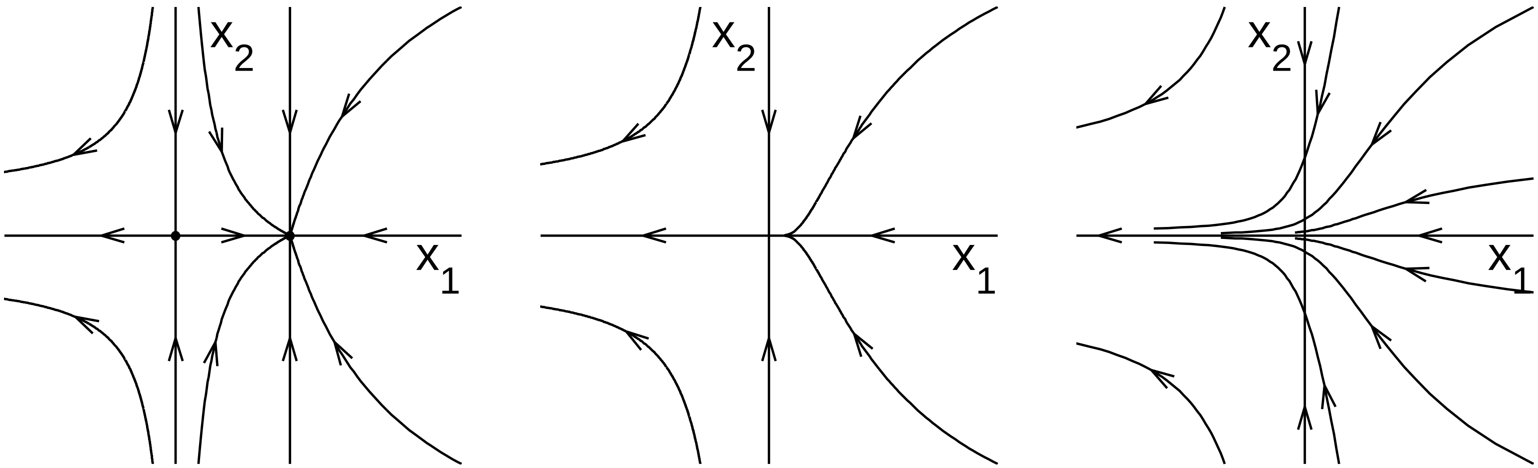
$$\begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

$(-\sqrt{\mu}, 0)$ is a saddle

$$\dot{x}_1 = \mu - x_1^2, \qquad \dot{x}_2 = -x_2$$

No equilibrium points when $\mu < 0$

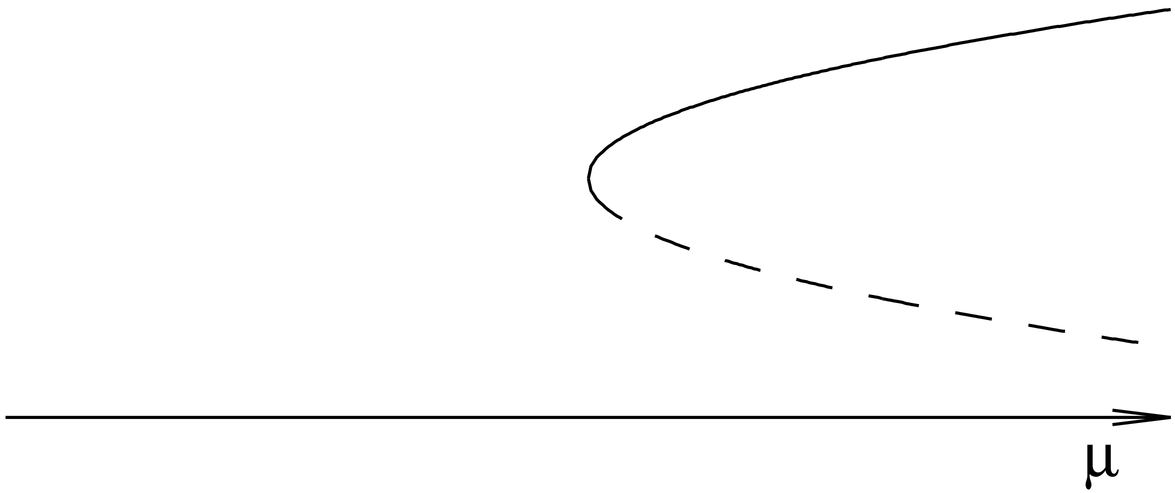
As μ decreases, the saddle and node approach each other, collide at $\mu = 0$, and disappear for $\mu < 0$



$\mu > 0$ $\mu = 0$ $\mu < 0$

μ is called the bifurcation parameter and $\mu = 0$ is the bifurcation point

Bifurcation Diagram



(a) Saddle−node bifurcation

Example

$$\dot{x}_1 = \mu x_1 - x_1^2, \qquad \dot{x}_2 = -x_2$$

Two equilibrium points at $(0, 0)$ and $(\mu, 0)$

The Jacobian at $(0, 0)$ is $\begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}$

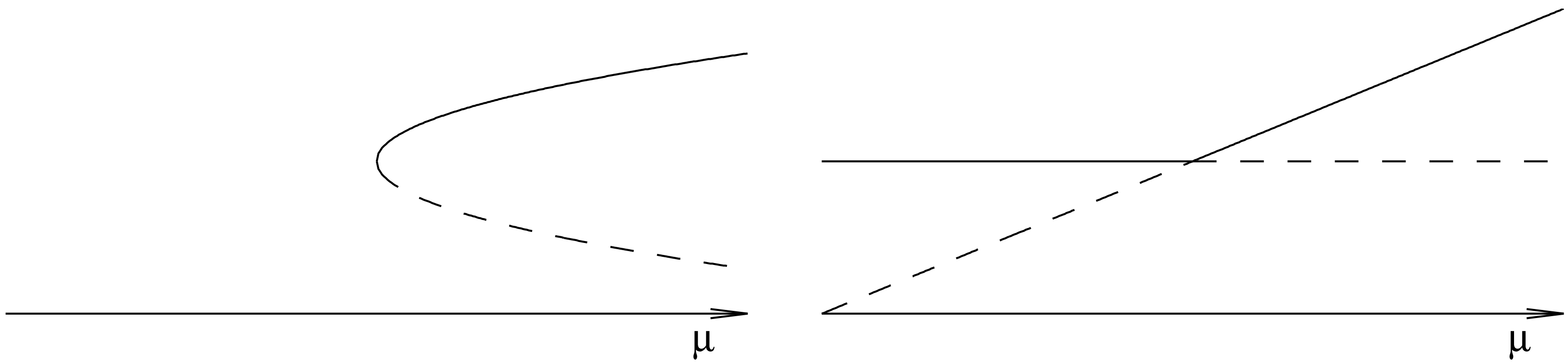
$(0, 0)$ is a stable node for $\mu < 0$ and a saddle for $\mu > 0$

The Jacobian at $(\mu, 0)$ is $\begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}$

$(\mu, 0)$ is a saddle for $\mu < 0$ and a stable node for $\mu > 0$

An eigenvalue crosses the origin as μ crosses zero

While the equilibrium points persist through the bifurcation point $\mu = 0$, $(0, 0)$ changes from a stable node to a saddle and $(\mu, 0)$ changes from a saddle to a stable node



(a) Saddle−node bifurcation

(b) Transcritical bifurcation

dangerous or hard

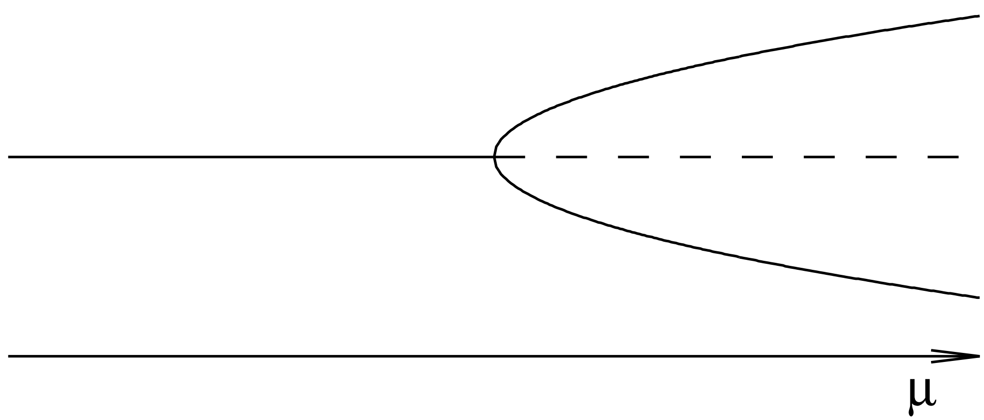
safe or soft

Example

$$\dot{x}_1 = \mu x_1 - x_1^3, \qquad \dot{x}_2 = -x_2$$

For $\mu < 0$, there is a stable node at the origin

For $\mu > 0$, there are three equilibrium points: a saddle at $(0,0)$ and stable nodes at $(\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$



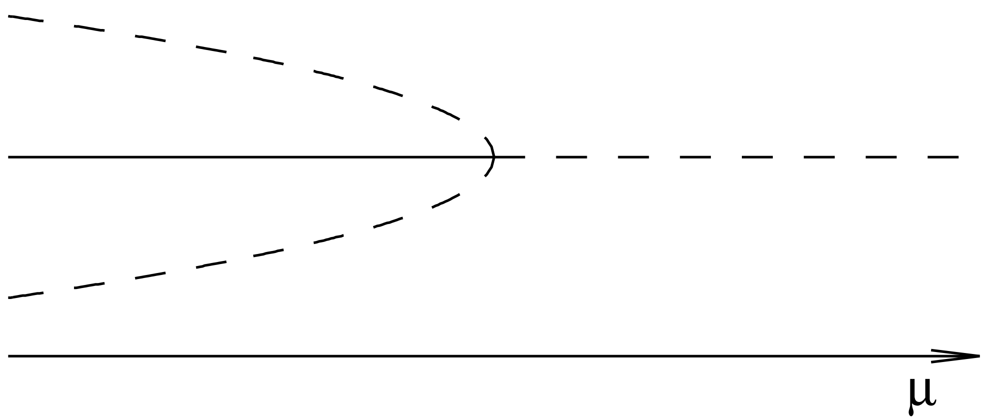
(c) Supercritical pitchfork bifurcation

Example

$$\dot{x}_1 = \mu x_1 + x_1^3, \qquad \dot{x}_2 = -x_2$$

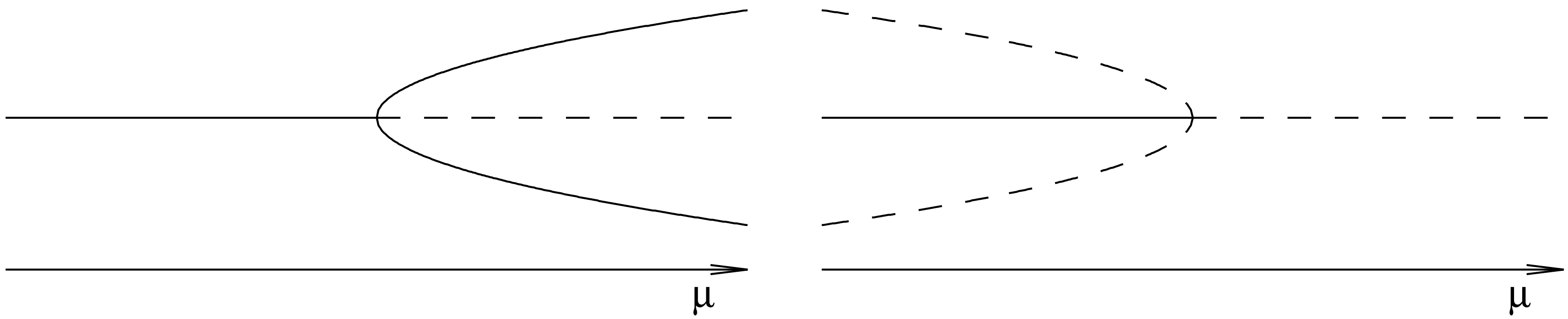
For $\mu < 0$, there are three equilibrium points: a stable node at $(0, 0)$ and two saddles at $(\pm\sqrt{-\mu}, 0)$

For $\mu > 0$, there is a saddle at $(0, 0)$



(d) Subcritical pitchfork bifurcation

Notice the difference between supercritical and subcritical pitchfork bifurcations



(c) Supercritical pitchfork bifurcation

(d) Subcritical pitchfork bifurcation

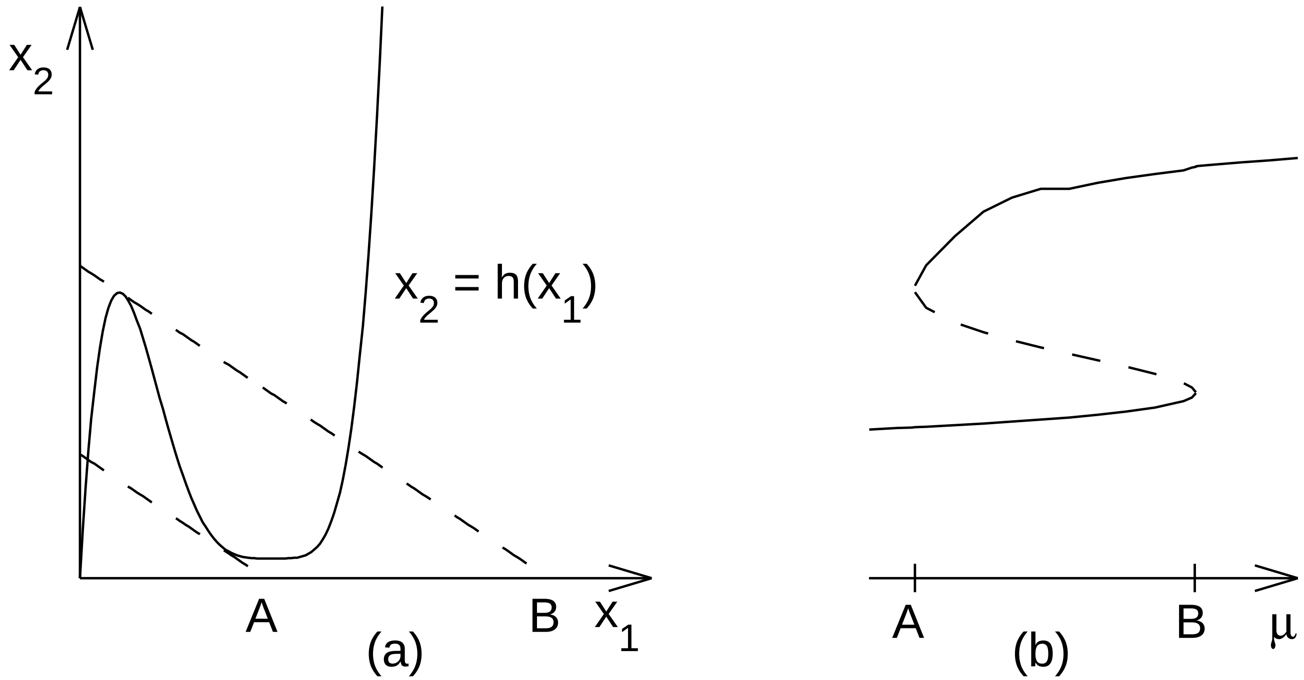
safe or soft

dangerous or hard

Example: Tunnel diode Circuit

$$\dot{x}_1 = \frac{1}{C}\left[-h(x_1) + x_2\right]$$

$$\dot{x}_2 = \frac{1}{L}\left[-x_1 - Rx_2 + \mu\right]$$



Example

$$\begin{aligned}\dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1\end{aligned}$$

There is a unique equilibrium point at the origin

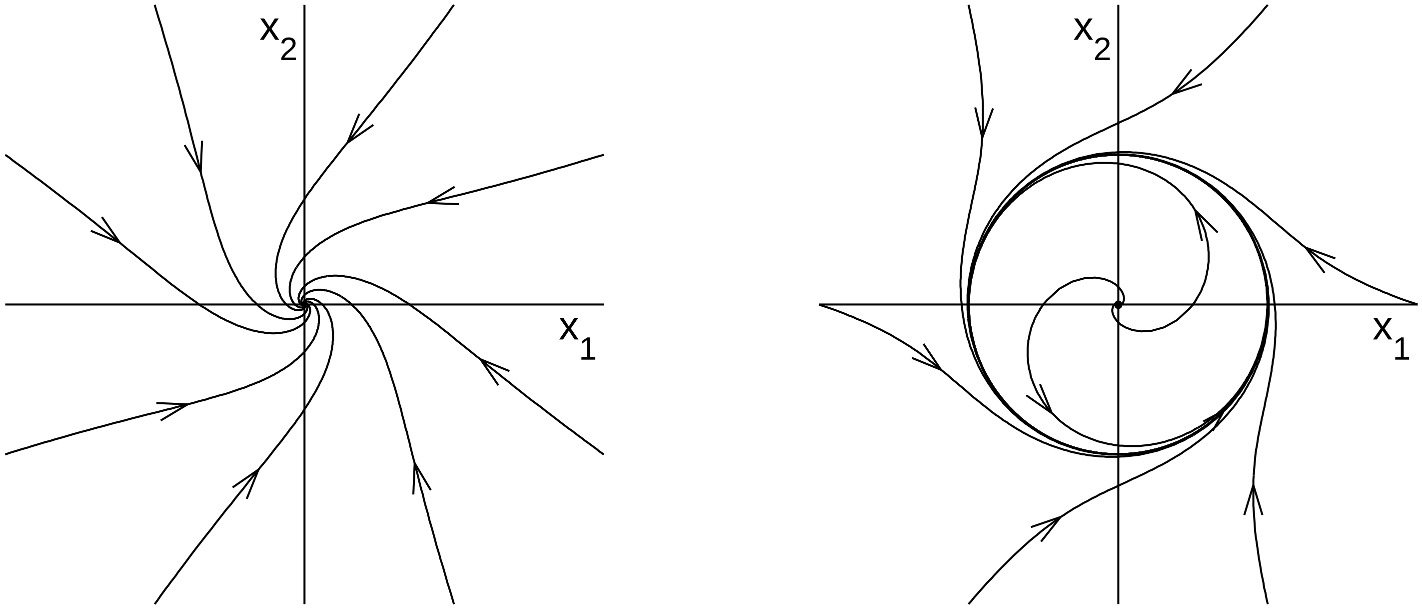
$$\text{Linearization:} \quad \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

Stable focus for $\mu < 0$, and unstable focus for $\mu > 0$

A pair of complex eigenvalues cross the imaginary axis as μ crosses zero

$$\dot{r} = \mu r - r^3 \quad \text{and} \quad \dot{\theta} = 1$$

For $\mu > 0$, there is a stable limit cycle at $r = \sqrt{\mu}$



$\mu < 0$ $\qquad$ $\mu > 0$

Supercritical Hopf bifurcation

Example

$$\begin{aligned}\dot{x}_1 &= x_1\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] - x_2\\ \dot{x}_2 &= x_2\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] + x_1\end{aligned}$$

There is a unique equilibrium point at the origin

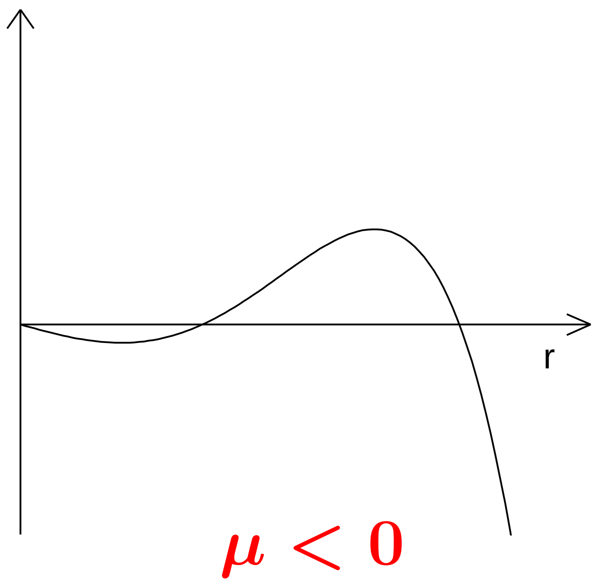
$$\text{Linearization:} \quad \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

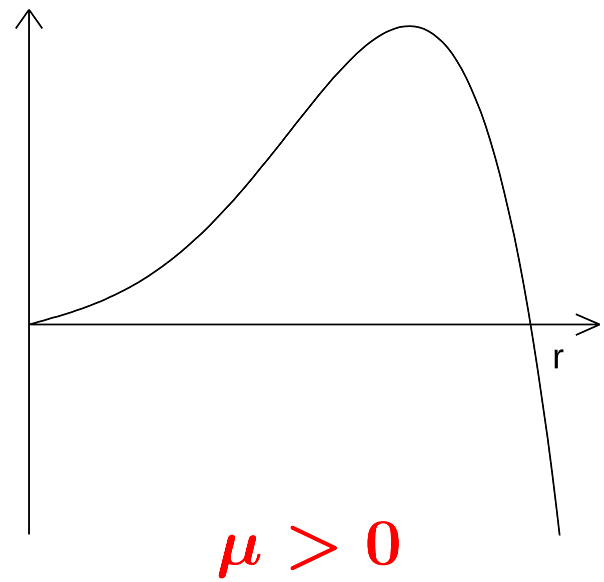
Stable focus for $\mu < 0$, and unstable focus for $\mu > 0$

A pair of complex eigenvalues cross the imaginary axis as μ crosses zero

$$\dot{r} = \mu r + r^3 - r^5 \quad \text{and} \quad \dot{\theta} = 1$$

Sketch of $\mu r + r^3 - r^5$:

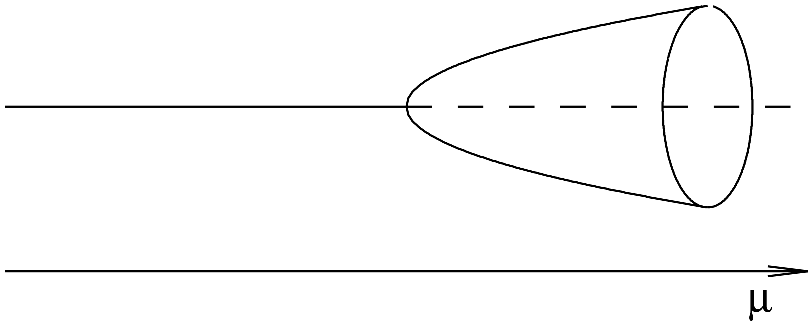




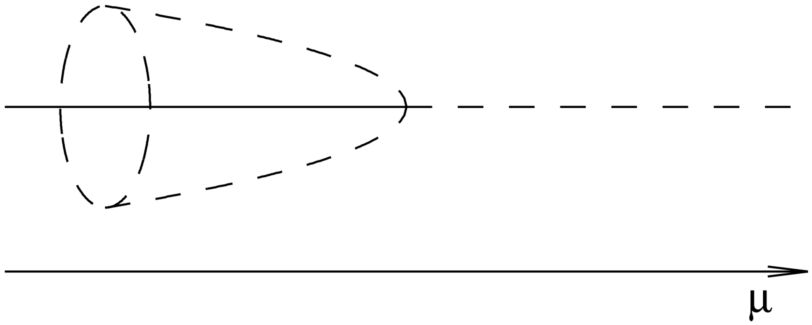
For small $|\mu|$, the stable limit cycles are approximated by $r = 1$, while the unstable limit cycle for $\mu < 0$ is approximated by $r = \sqrt{|\mu|}$

As μ increases from negative to positive values, the stable focus at the origin merges with the unstable limit cycle and bifurcates into unstable focus

Subcritical Hopf bifurcation



(e) Supercritical Hopf bifurcation



(f) Subcrtitical Hopf bifurcation

safe or soft

dangerous or hard

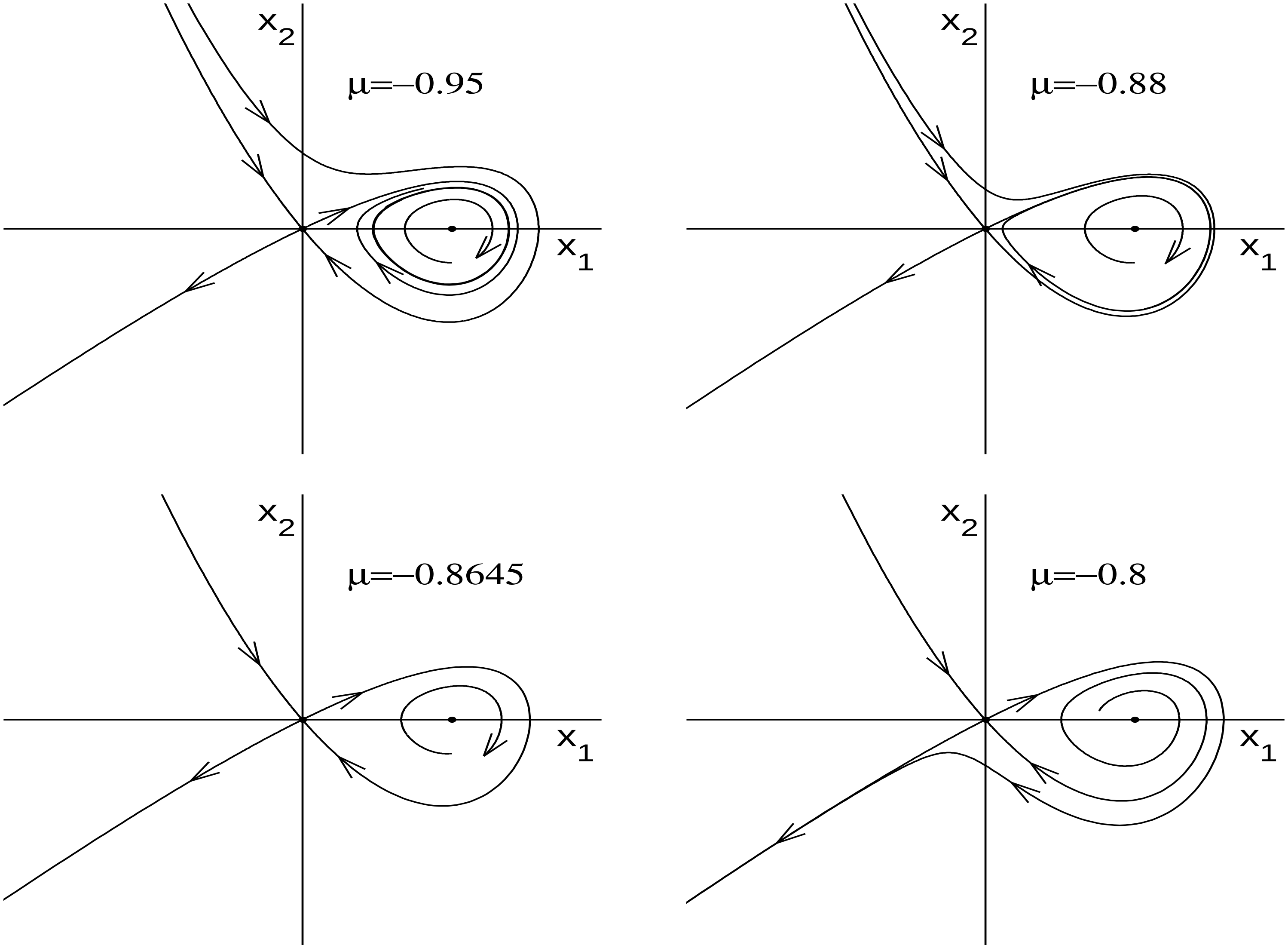
All six types of bifurcation occur in the vicinity of an equilibrium point. They are called local bifurcations

Example of Global Bifurcation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu x_2 + x_1 - x_1^2 + x_1 x_2\end{aligned}$$

There are two equilibrium points at $(0, 0)$ and $(1, 0)$. By linearization, we can see that $(0, 0)$ is always a saddle, while $(1, 0)$ is an unstable focus for $-1 < \mu < 1$

Limit analysis to the range $-1 < \mu < 1$



Saddle–connection (or homoclinic) bifurcation

Nonlinear Systems and Control
Lecture # 7
Stability of Equilibrium Points
Basic Concepts & Linearization

$$\dot{x} = f(x)$$

f is locally Lipschitz over a domain $D \subset R^n$

Suppose $\bar{x} \in D$ is an equilibrium point; that is, $f(\bar{x}) = 0$

Characterize and study the stability of $\bar{x}$

For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. No loss of generality

$$y = x - \bar{x}$$

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- stable if for each $\varepsilon > 0$ there is $\delta > 0$ (dependent on ε) such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \ \forall \, t \geq 0$$

- unstable if it is not stable

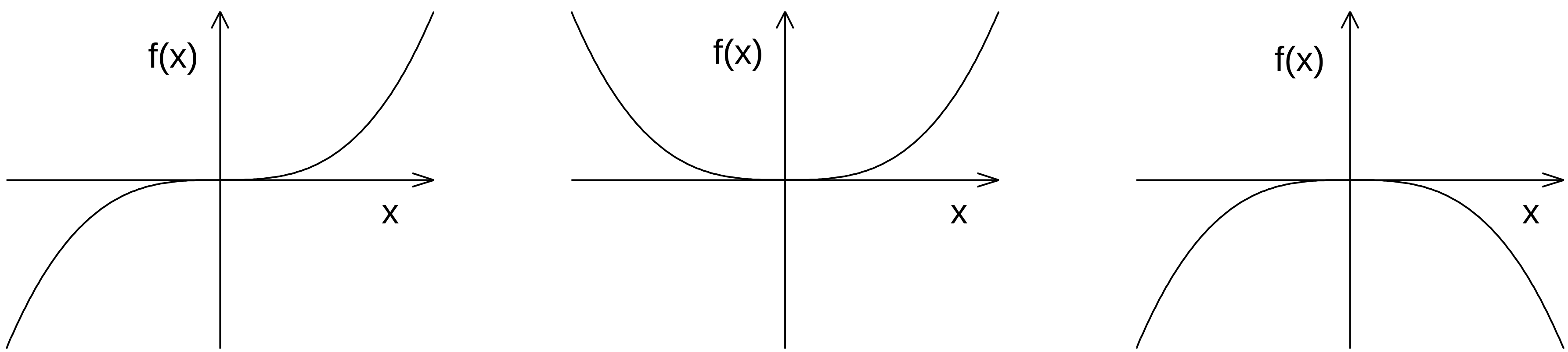
- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

First-Order Systems $(n = 1)$

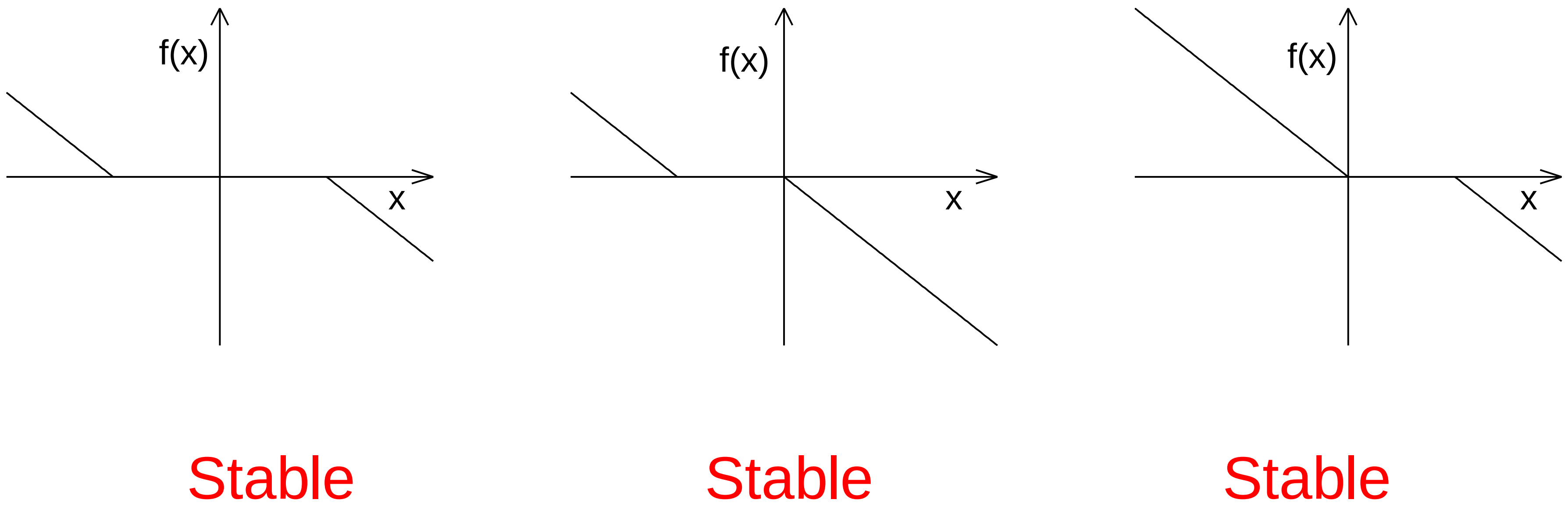
The behavior of $x(t)$ in the neighborhood of the origin can be determined by examining the sign of $f(x)$

The ε–δ requirement for stability is violated if $xf(x) > 0$ on either side of the origin

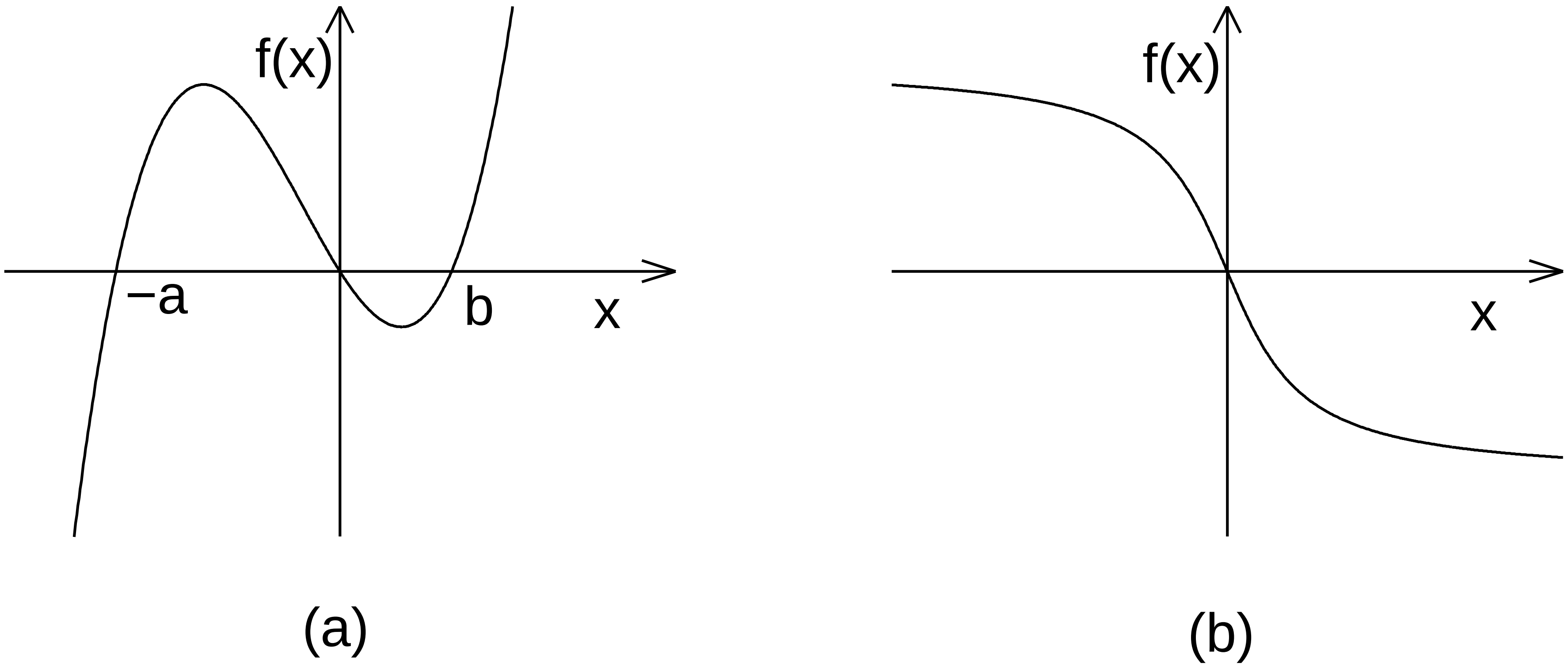


Unstable Unstable Unstable

The origin is stable if and only if $xf(x) \leq 0$ in some neighborhood of the origin



The origin is asymptotically stable if and only if $xf(x) < 0$ in some neighborhood of the origin



Asymptotically Stable

Globally Asymptotically Stable

Definition: Let the origin be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where f is a locally Lipschitz function defined over a domain $D \subset R^n$ ($0 \in D$)

- The region of attraction (also called region of asymptotic stability, domain of attraction, or basin) is the set of all points x_0 in D such that the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

  is defined for all $t \geq 0$ and converges to the origin as t tends to infinity

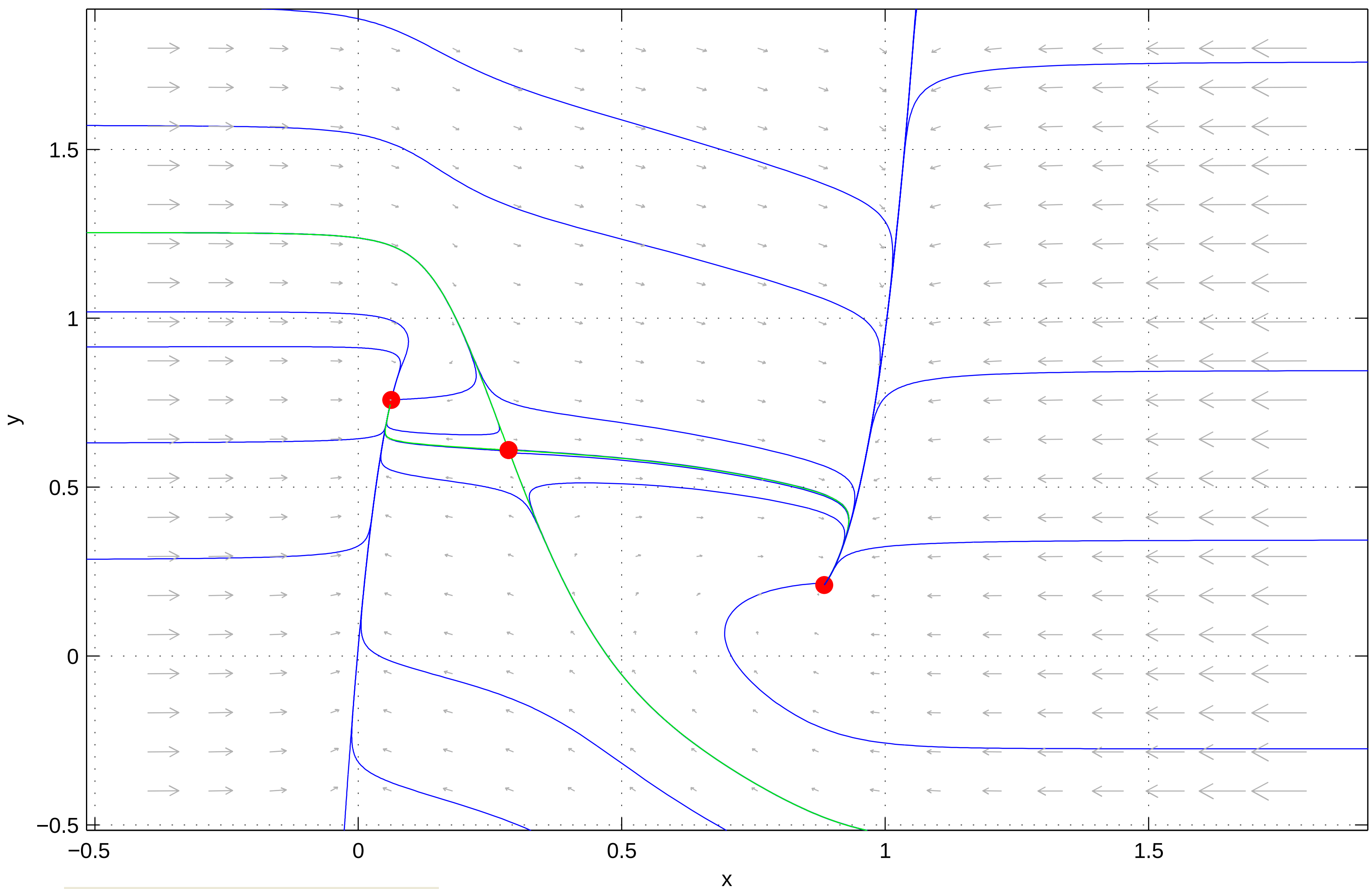
- The origin is said to be globally asymptotically stable if the region of attraction is the whole space R^n

Second-Order Systems $(n = 2)$

Type of equilibrium point	Stability Property
Center	
Stable Node	
Stable Focus	
Unstable Node	
Unstable Focus	
Saddle	

Example: Tunnel Diode Circuit

$x' = 0.5\,(-17.76\,x + 103.79\,x^2 - 229.62\,x^3 + 226.31\,x^4 - 83.72\,x^5 + y)$
$y' = 0.2\,(-x - 1.5\,y + 1.2)$



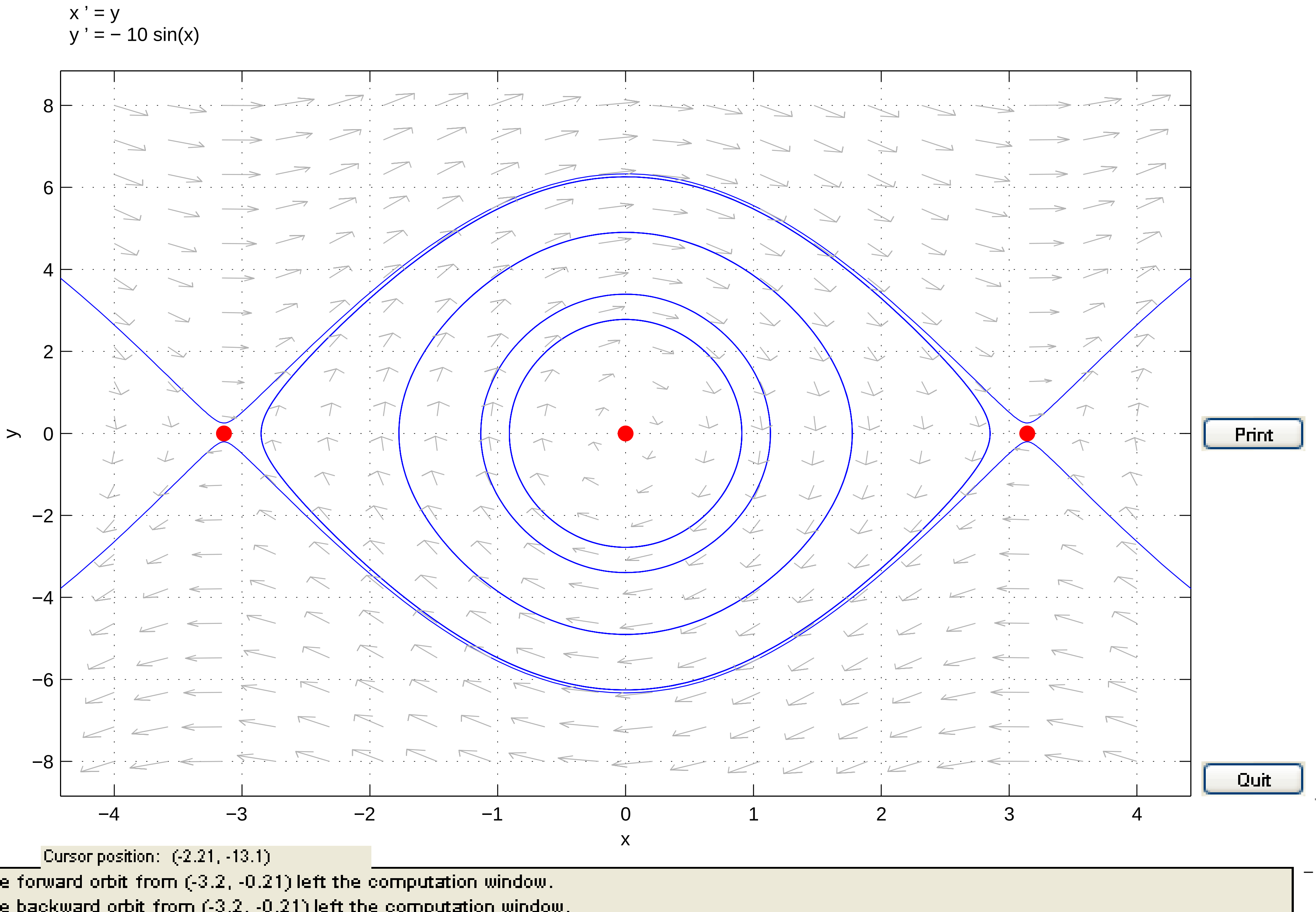
Print

Quit

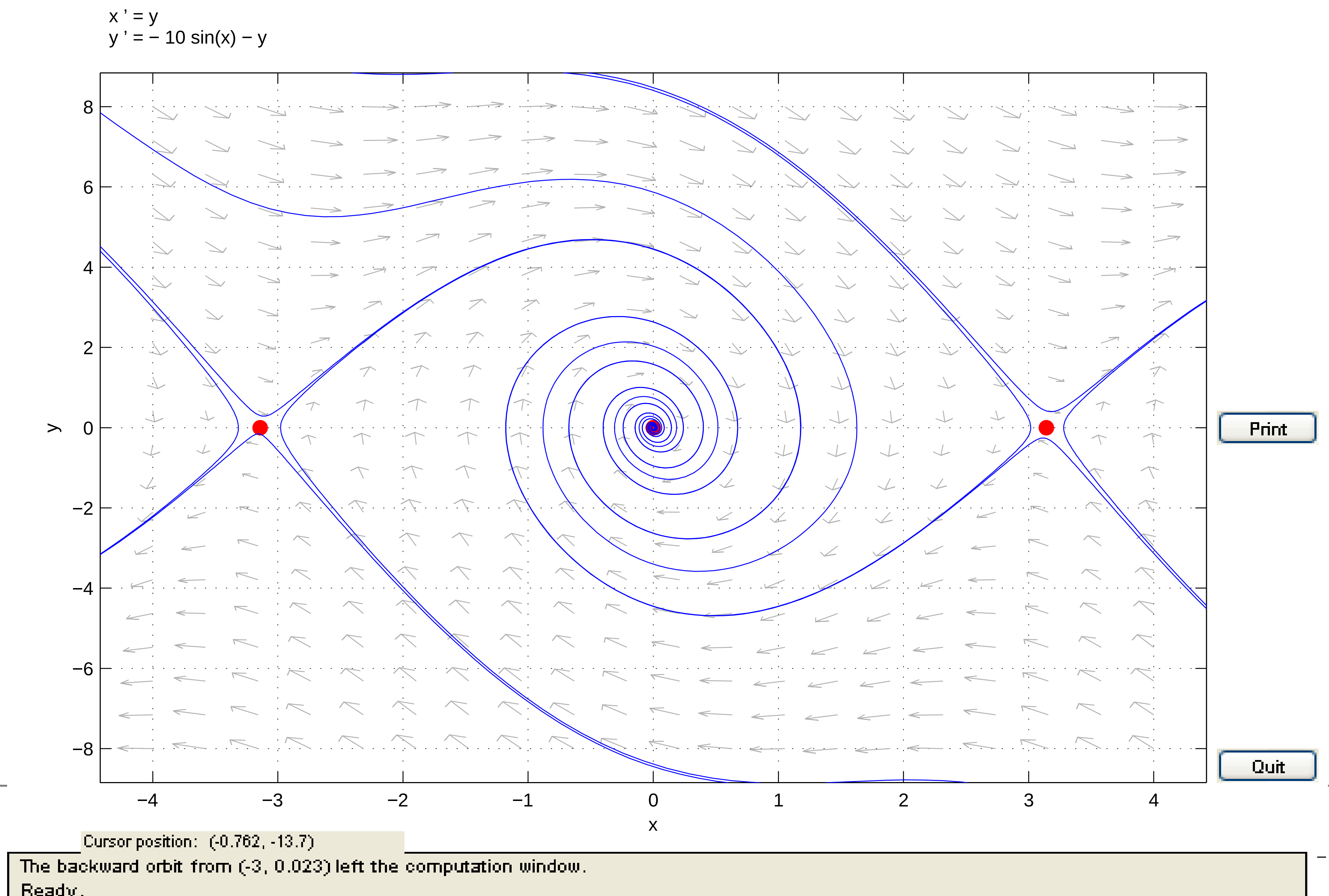
Cursor position: (1.02, -0.908)

The second unstable trajectory --> a possible eq. pt. near (0.063, 0.76).
Ready

Example: Pendulum Without Friction



Example: Pendulum With Friction



Linear Time-Invariant Systems

$$\dot{x} = Ax$$

$$x(t) = \exp(At)x(0)$$

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\lambda_i t)R_{ik}$$

m_i is the order of the Jordan block J_i

$$\mathrm{Re}[\lambda_i] < 0 \ \forall\, i \quad \Leftrightarrow \quad \text{Asymptotically Stable}$$

$$\mathrm{Re}[\lambda_i] > 0 \ \text{ for some } i \quad \Rightarrow \quad \text{Unstable}$$

$$\mathrm{Re}[\lambda_i] \le 0 \ \forall\, i \ \&\ m_i > 1 \text{ for } \mathrm{Re}[\lambda_i] = 0 \ \Rightarrow \quad \text{Unstable}$$

$$\mathrm{Re}[\lambda_i] \le 0 \ \forall\, i \ \&\ m_i = 1 \text{ for } \mathrm{Re}[\lambda_i] = 0 \ \Rightarrow \quad \text{Stable}$$

If an $n \times n$ matrix A has a repeated eigenvalue λ_i of algebraic multiplicity q_i, then the Jordan blocks of λ_i have order one if and only if $\mathrm{rank}(A - \lambda_i I) = n - q_i$

Theorem: The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \leq 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$

When all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$, A is called a *Hurwitz matrix*

When the origin of a linear system is asymptotically stable, its solution satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0$$

$k \geq 1, \ \lambda > 0$

Exponential Stability

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is said to be exponentially stable if

$$\|x(t)\| \le k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \ge 0$$

$k \ge 1,\ \lambda > 0$, for all $\|x(0)\| < c$

It is said to be globally exponentially stable if the inequality is satisfied for any initial state $x(0)$

Exponential Stability $\Rightarrow$ Asymptotic Stability

Example

$$\dot{x} = -x^3$$

The origin is asymptotically stable

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$

$x(t)$ does not satisfy $|x(t)| \leq ke^{-\lambda t}|x(0)|$ because

$$|x(t)| \leq ke^{-\lambda t}|x(0)| \;\Rightarrow\; \frac{e^{2\lambda t}}{1 + 2tx^2(0)} \leq k^2$$

Impossible because $\displaystyle \lim_{t\to\infty} \frac{e^{2\lambda t}}{1 + 2tx^2(0)} = \infty$

Linearization

$$\dot{x} = f(x), \quad f(0) = 0$$

f is continuously differentiable over $D = \{\|x\| < r\}$

$$J(x) = \frac{\partial f}{\partial x}(x)$$

$$h(\sigma) = f(\sigma x) \text{ for } 0 \leq \sigma \leq 1$$

$$h'(\sigma) = J(\sigma x)x$$

$$h(1) - h(0) = \int_0^1 h'(\sigma)\, d\sigma, \quad h(0) = f(0) = 0$$

$$f(x) = \int_0^1 J(\sigma x)\, d\sigma\ x$$

$$f(x) = \int_0^1 J(\sigma x)\, d\sigma\, x$$

Set $A = J(0)$ and add and subtract Ax

$$f(x) = [A + G(x)]x, \text{ where } G(x) = \int_0^1 [J(\sigma x) - J(0)]\, d\sigma$$

$$G(x) \to 0 \ \text{ as } \ x \to 0$$

This suggests that in a small neighborhood of the origin we can approximate the nonlinear system $\dot{x} = f(x)$ by its linearization about the origin $\dot{x} = Ax$

Theorem:

- The origin is exponentially stable if and only if $\mathrm{Re}[\lambda_i] < 0$ for all eigenvalues of A
- The origin is unstable if $\mathrm{Re}[\lambda_i] > 0$ for some i

Linearization fails when $\mathrm{Re}[\lambda_i] \leq 0$ for all i, with $\mathrm{Re}[\lambda_i] = 0$ for some i

Example

$$\dot{x} = ax^3$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\Big|_{x=0} = 0$$

Stable if $a = 0$; Asymp stable if $a < 0$; Unstable if $a > 0$

When $a < 0$, the origin is not exponentially stable

Nonlinear Systems and Control
Lecture # 8
Lyapunov Stability

Let $V(x)$ be a continuously differentiable function defined in a domain $D \subset R^n$; $0 \in D$. The derivative of V along the trajectories of $\dot{x} = f(x)$ is

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\ &= \left[\begin{array}{cccc} \frac{\partial V}{\partial x_1}, & \frac{\partial V}{\partial x_2}, & \cdots, & \frac{\partial V}{\partial x_n} \end{array} \right] \left[\begin{array}{c} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{array} \right] \\ &= \frac{\partial V}{\partial x} f(x) \end{aligned}$$

If $\phi(t;x)$ is the solution of $\dot{x} = f(x)$ that starts at initial state x at time $t = 0$, then

$$\dot{V}(x) = \frac{d}{dt} V(\phi(t;x))\Big|_{t=0}$$

If $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$

If $\dot{V}(x)$ is positive, V will increase along the solution of $\dot{x} = f(x)$

Lyapunov's Theorem:

- If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \in D/\{0\}$$

$$\dot{V}(x) \leq 0, \quad \forall\, x \in D$$

then the origin is a stable

- Moreover, if

$$\dot{V}(x) < 0, \quad \forall\, x \in D/\{0\}$$

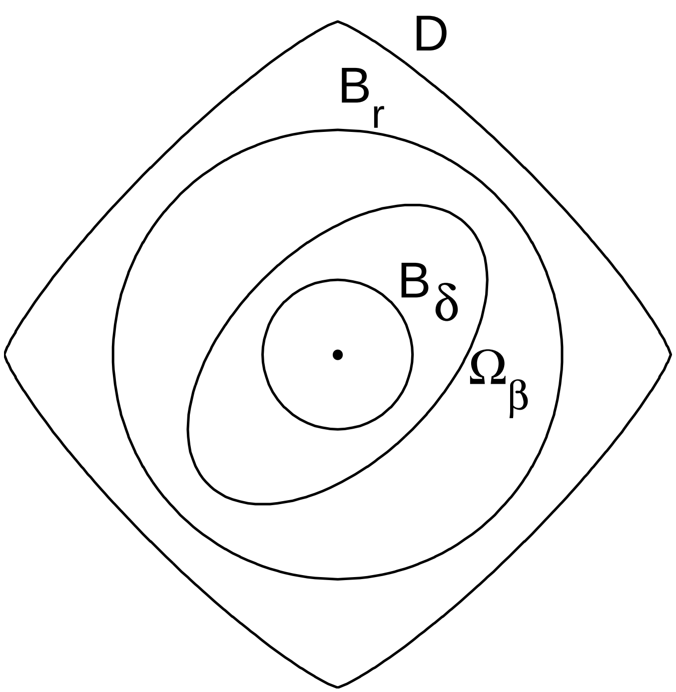
then the origin is asymptotically stable

- Furthermore, if $V(x) > 0$, $\forall\, x \neq 0$,

$$\|x\| \rightarrow \infty \;\Rightarrow\; V(x) \rightarrow \infty$$

and $\dot{V}(x) < 0$, $\forall\, x \neq 0$, then the origin is globally asymptotically stable

Proof:



$$0 < r \leq \varepsilon,\;\; B_r = \{\|x\| \leq r\}$$

$$\alpha = \min_{\|x\|=r} V(x) > 0$$

$$0 < \beta < \alpha$$

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

$$\|x\| \leq \delta \;\Rightarrow\; V(x) < \beta$$

Solutions starting in Ω_β stay in Ω_β because $\dot{V}(x) \leq 0$ in Ω_β

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \ \ \forall\, t \geq 0$$

$\Rightarrow$ The origin is stable

Now suppose $\dot{V}(x) < 0 \; \forall\, x \in D/\{0\}$. $V(x(t))$ is monotonically decreasing and $V(x(t)) \geq 0$

$$\lim_{t\to\infty} V(x(t)) = c \geq 0$$

$$\lim_{t\to\infty} V(x(t)) = c \geq 0 \quad \text{Show that } c = 0$$

Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. Then, $x(t)$ lies outside B_d for all $t \geq 0$

$$\gamma = -\max_{d \le \|x\| \le r} \dot{V}(x)$$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \le V(x(0)) - \gamma t$$

This inequality contradicts the assumption $c > 0$

$\Rightarrow$ The origin is asymptotically stable

The condition $\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty$ implies that the set $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is compact for every $c > 0$. This is so because for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$. All solutions starting Ω_c will converge to the origin. For any point $p \in R^n$, choosing $c = V(p)$ ensures that $p \in \Omega_c$

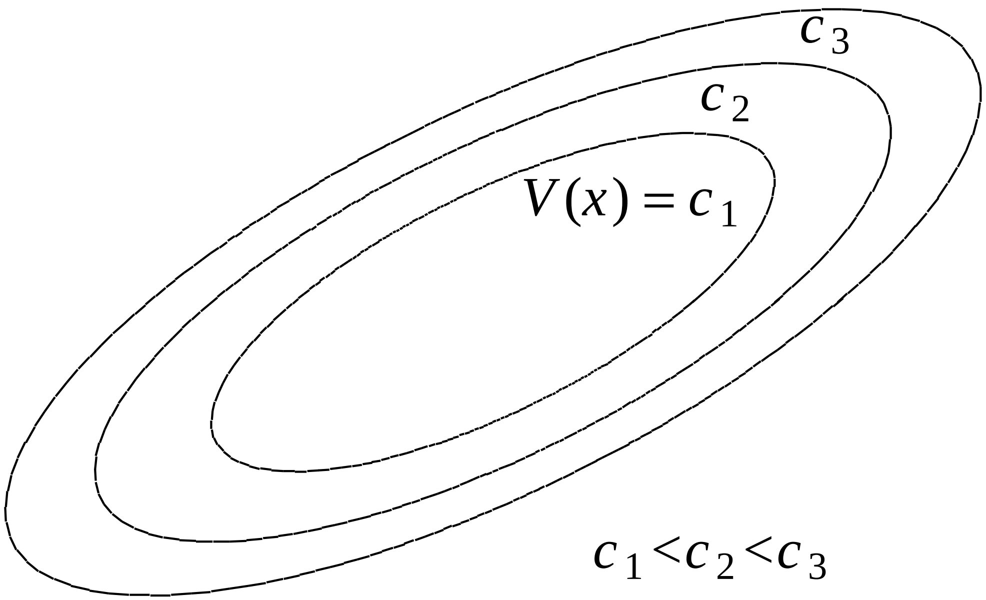
$\Rightarrow$ The origin is globally asymptotically stable

Terminology

$V(0) = 0,\ V(x) \geq 0$ for $x \neq 0$	Positive semidefinite
$V(0) = 0,\ V(x) > 0$ for $x \neq 0$	Positive definite
$V(0) = 0,\ V(x) \leq 0$ for $x \neq 0$	Negative semidefinite
$V(0) = 0,\ V(x) < 0$ for $x \neq 0$	Negative definite
$\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty$	Radially unbounded

Lyapunov' Theorem: The origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite. It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded

A continuously differentiable function $V(x)$ satisfying the conditions for stability is called a *Lyapunov function*. The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a *level surface*



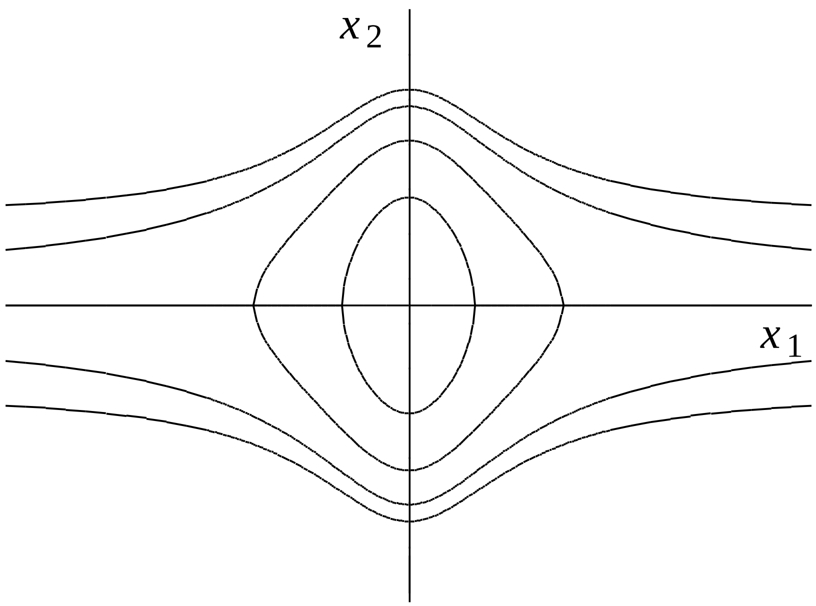
Why do we need the radial unboundedness condition to show global asymptotic stability?

It ensures that $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is bounded for every $c > 0$

Without it Ω_c might not bounded for large c

Example

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$



Nonlinear Systems and Control
Lecture # 9
Lyapunov Stability

Quadratic Forms

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \quad P = P^T$$

$$\lambda_{\min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|^2$$

$P \geq 0$ (Positive semidefinite) if and only if $\lambda_i(P) \geq 0 \; \forall i$

$P > 0$ (Positive definite) if and only if $\lambda_i(P) > 0 \; \forall i$

$V(x)$ is positive definite if and only if P is positive definite

$V(x)$ is positive semidefinite if and only if P is positive semidefinite

$P > 0$ if and only if all the leading principal minors of P are positive

Linear Systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x, \qquad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x \stackrel{\text{def}}{=} -x^T Q x$$

If $Q > 0$, then A is Hurwitz

Or choose $Q > 0$ and solve the Lyapunov equation

$$PA + A^T P = -Q$$

If $P > 0$, then A is Hurwitz

Matlab: $P = \text{lyap}(A', Q)$

Theorem A matrix A is Hurwitz if and only if for any $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

Idea of the proof: Sufficiency follows from Lyapunov's theorem. Necessity is shown by verifying that

$$P = \int_0^\infty \exp(A^T t) Q \exp(At) \, dt$$

is positive definite and satisfies the Lyapunov equation

Linearization

$$\dot{x} = f(x) = [A + G(x)]x$$

$$G(x) \longrightarrow 0 \;\; \text{as} \;\; x \longrightarrow 0$$

Suppose A is Hurwitz. Choose $Q = Q^T > 0$ and solve the Lyapunov equation $PA + A^TP = -Q$ for P. Use $V(x) = x^TPx$ as a Lyapunov function candidate for $\dot{x} = f(x)$

$$\begin{aligned}
\dot{V}(x) &= x^TPf(x) + f^T(x)Px \\
&= x^TP[A + G(x)]x + x^T[A^T + G^T(x)]Px \\
&= x^T(PA + A^TP)x + 2x^TPG(x)x \\
&= -x^TQx + 2x^TPG(x)x
\end{aligned}$$

$$\dot{V}(x) \le -x^T Q x + 2\|P\| \, \|G(x)\| \, \|x\|^2$$

For any $\gamma > 0$, there exists $r > 0$ such that

$$\|G(x)\| < \gamma, \;\; \forall \; \|x\| < r$$

$$x^T Q x \ge \lambda_{\min}(Q)\|x\|^2 \;\Leftrightarrow\; -x^T Q x \le -\lambda_{\min}(Q)\|x\|^2$$

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma\|P\|]\|x\|^2, \;\; \forall \; \|x\| < r$$

Choose

$$\gamma < \frac{\lambda_{\min}(Q)}{2\|P\|}$$

$V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = f(x)$

We can use $V(x) = x^T P x$ to estimate the region of attraction

$$\text{Suppose} \quad \dot{V}(x) < 0, \quad \forall\, 0 < \|x\| < r$$

$$\text{Take} \quad c = \min_{\|x\|=r} x^T P x = \lambda_{\min}(P) r^2$$

$$\{x^T P x < c\} \subset \{\|x\| < r\}$$

All trajectories starting in the set $\{x^T P x < c\}$ approach the origin as t tends to ∞. Hence, the set $\{x^T P x < c\}$ is a subset of the region of attraction (an estimate of the region of attraction)

Example

$$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

has eigenvalues $(-1 \pm j\sqrt{3})/2$. Hence the origin is asymptotically stable

Take $Q = I, \quad PA + A^T P = -I \;\Rightarrow\; P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

$\lambda_{\min}(P) = 0.691$

$$V(x) = x^T P x = 1.5x_1^2 - x_1 x_2 + x_2^2$$

$$\begin{aligned} \dot{V}(x) &= (3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)[x_1 + (x_1^2 - 1)x_2] \\ &= -(x_1^2 + x_2^2) - (x_1^3 x_2 - 2x_1^2 x_2^2) \end{aligned}$$

$$\dot{V}(x) \le -\|x\|^2 + |x_1|\,|x_1 x_2|\,|x_1 - 2x_2| \le -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4$$

where $|x_1| \le \|x\|, \ |x_1 x_2| \le \frac{1}{2}\|x\|^2, \ |x_1 - 2x_2| \le \sqrt{5}\|x\|$

$$\dot{V}(x) < 0 \ \text{ for } 0 < \|x\|^2 < \frac{2}{\sqrt{5}} \stackrel{\text{def}}{=} r^2$$

Take $c = \lambda_{\min}(P) r^2 = 0.691 \times \dfrac{2}{\sqrt{5}} = 0.618$

$\{V(x) < c\}$ is an estimate of the region of attraction

Example:

$$\dot{x} = -g(x)$$

$$g(0) = 0; \;\; xg(x) > 0, \;\; \forall\, x \neq 0 \;\text{ and }\; x \in (-a, a)$$

$$V(x) = \int_0^x g(y)\, dy$$

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \; \forall\, x \in (-a, a), \; x \neq 0$$

The origin is asymptotically stable

If $xg(x) > 0$ for all $x \neq 0$, use

$$V(x) = \tfrac{1}{2}x^2 + \int_0^x g(y)\, dy$$

$$V(x) = \frac{1}{2}x^2 + \int_0^x g(y)\, dy$$

is positive definite for all x and radially unbounded since $V(x) \geq \frac{1}{2}x^2$

$$\dot{V}(x) = -xg(x) - g^2(x) < 0, \quad \forall\, x \neq 0$$

The origin is globally asymptotically stable

Example: Pendulum equation without friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \tfrac{1}{2}x_2^2$$

$V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0$$

The origin is stable

Since $\dot{V}(x) \equiv 0$, the origin is not asymptotically stable

Example: Pendulum equation with friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - bx_2 \end{aligned}$$

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

The origin is stable

$\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

The conditions of Lyapunov's theorem are only sufficient. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate

Try

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T P x + a(1-\cos x_1) \\ &= \tfrac{1}{2}[x_1\ x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1-\cos x_1) \end{aligned}$$

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

$$\begin{aligned}\dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a\sin x_1)\,x_2 \\ &\quad + (p_{12}x_1 + p_{22}x_2)\,(-a\sin x_1 - bx_2) \\ &= a(1-p_{22})x_2\sin x_1 - ap_{12}x_1\sin x_1 \\ &\quad + (p_{11} - p_{12}b)\,x_1x_2 + (p_{12} - p_{22}b)\,x_2^2\end{aligned}$$

$p_{22} = 1, \quad p_{11} = bp_{12} \;\Rightarrow\; 0 < p_{12} < b,$ Take $p_{12} = b/2$

$$\dot{V}(x) = -\tfrac{1}{2}abx_1\sin x_1 - \tfrac{1}{2}bx_2^2$$

$$D = \{x \in R^2 \mid |x_1| < \pi\}$$

$V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D
The origin is asymptotically stable

Read about the variable gradient method in the textbook

Nonlinear Systems and Control
Lecture # 10
The Invariance Principle

Example: Pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - bx_2\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

The origin is stable. $\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

However, near the origin, the solution cannot stay identically in the set $\{x_2 = 0\}$

Definitions: Let $x(t)$ be a solution of $\dot{x} = f(x)$

A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $\lim_{n\to\infty} t_n = \infty$, such that $x(t_n) \to p$ as $n \to \infty$

The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$; denoted by L^+

If $x(t)$ approaches an asymptotically stable equilibrium point $\bar{x}$, then $\bar{x}$ is the positive limit point of $x(t)$ and $L^+ = \bar{x}$

A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle

A set M is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \ \ \forall\, t \in R$$

Examples:

- Equilibrium points
- Limit Cycles

A set M is a *positively invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \ \ \forall\, t \geq 0$$

Example: The set $\Omega_c = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c

The distance from a point p to a set M is defined by

$$\mathrm{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

$x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\mathrm{dist}(x(t), M) < \varepsilon, \quad \forall\, t > T$$

Example: every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \to \infty$

Notice, however, that $x(t)$ does converge to any specific point on the limit cycle

Lemma: If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$

LaSalle's theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$

Proof:

$$\dot{V}(x) \leq \text{ in } \Omega \;\Rightarrow\; V(x(t)) \text{ is a decreasing}$$

$$V(x) \text{ is continuous in } \Omega \;\Rightarrow\; V(x) \geq b = \min_{x \in \Omega} V(x)$$

$$\Rightarrow \lim_{t \to \infty} V(x(t)) = a$$

$$x(t) \in \Omega \;\Rightarrow\; x(t) \text{ is bounded} \;\Rightarrow\; L^+ \text{ exists}$$

Moreover, $L^+ \subset \Omega$ and $x(t)$ approaches L^+ as $t \to \infty$

For any $p \in L^+$, there is $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $x(t_n) \to p$ as $n \to \infty$

$$V(x) \text{ is continuous} \;\Rightarrow\; V(p) = \lim_{n \to \infty} V(x(t_n)) = a$$

$\boldsymbol{V(x) = a}$ on $\boldsymbol{L^+}$ and $\boldsymbol{L^+}$ invariant $\Rightarrow$ $\boldsymbol{\dot{V}(x) = 0}$, $\forall\, \boldsymbol{x} \in \boldsymbol{L^+}$

$$\boldsymbol{L^+ \subset M \subset E \subset \Omega}$$

$\boldsymbol{x(t)}$ approaches $\boldsymbol{L^+}$ $\Rightarrow$ $\boldsymbol{x(t)}$ approaches $\boldsymbol{M}$ (as $\boldsymbol{t \to \infty}$)

Theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable
- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction
- Furthermore, if $D = R^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2)\end{aligned}$$

$$h_i(0) = 0, \quad y h_i(y) > 0, \quad \text{for } 0 < |y| < a$$

$$V(x) = \int_0^{x_1} h_1(y)\, dy \; + \; \tfrac{1}{2}x_2^2$$

$$D = \{-a < x_1 < a, \;\; -a < x_2 < a\}$$

$$\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \le 0$$

$$\dot{V}(x) = 0 \;\Rightarrow\; x_2 h_2(x_2) = 0 \;\Rightarrow\; x_2 = 0$$

$$S = \{x \in D \mid x_2 = 0\}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

The only solution that can stay identically in S is $x(t) \equiv 0$

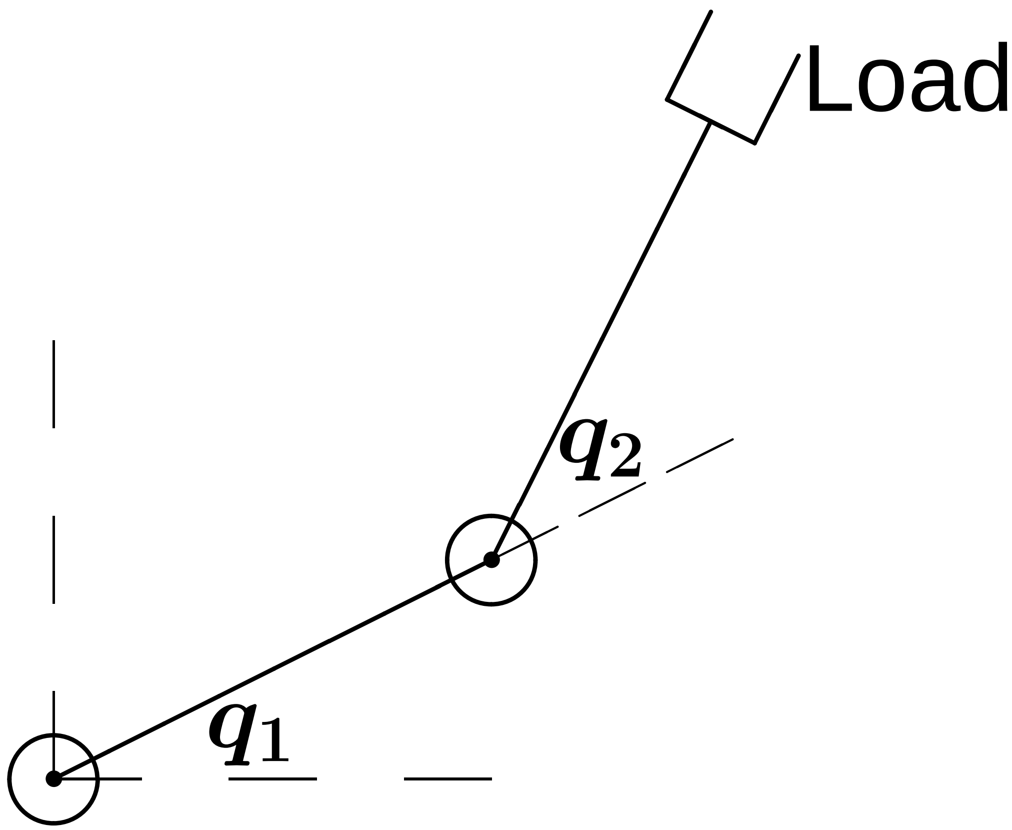
Thus, the origin is asymptotically stable

Suppose $a = \infty$ and $\int_0^y h_1(z)\, dz \; \to \; \infty \;$ as $\; |y| \to \; \infty$

Then, $D = R^2$ and $V(x) = \int_0^{x_1} h_1(y)\, dy \; + \frac{1}{2}x_2^2$ is radially unbounded. $S = \{x \in R^2 \mid x_2 = 0\}$ and the only solution that can stay identically in S is $x(t) \equiv 0$

The origin is globally asymptotically stable

Example: m-link Robot Manipulator



Two-link Robot Manipulator

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

q is an m-dimensional vector of joint positions

u is an m-dimensional control (torque) inputs

$M = M^T > 0$ is the inertia matrix

$C(q,\dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces

$$(\dot{M} - 2C)^T = -(\dot{M} - 2C)$$

$D\dot{q}$ accounts for viscous damping; $D = D^T \geq 0$

$g(q)$ accounts for gravity forces; $g(q) = [\partial P(q)/\partial q]^T$

$P(q)$ is the total potential energy of the links due to gravity

Investigate the use of the (PD plus gravity compensation) control law

$$\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{K}_p(\boldsymbol{q} - \boldsymbol{q}^*) - \boldsymbol{K}_d\,\dot{\boldsymbol{q}}$$

to stabilize the robot at a desired position $\boldsymbol{q}^*$, where $\boldsymbol{K}_p$ and $\boldsymbol{K}_d$ are symmetric positive definite matrices

$$\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}^*, \quad \dot{\boldsymbol{e}} = \dot{\boldsymbol{q}}$$

$$\begin{aligned}
\boldsymbol{M}\ddot{\boldsymbol{e}} &= \boldsymbol{M}\ddot{\boldsymbol{q}} \\
&= -\boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{D}\,\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{u} \\
&= -\boldsymbol{C}\,\dot{\boldsymbol{q}} - \boldsymbol{D}\,\dot{\boldsymbol{q}} - \boldsymbol{K}_p(\boldsymbol{q} - \boldsymbol{q}^*) - \boldsymbol{K}_d\,\dot{\boldsymbol{q}} \\
&= -\boldsymbol{C}\,\dot{\boldsymbol{e}} - \boldsymbol{D}\,\dot{\boldsymbol{e}} - \boldsymbol{K}_p\,\boldsymbol{e} - \boldsymbol{K}_d\,\dot{\boldsymbol{e}}
\end{aligned}$$

$$\boldsymbol{M\ddot{e} = -C\,\dot{e} - D\,\dot{e} - K_p\,e - K_d\,\dot{e}}$$

$$\boldsymbol{V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e}$$

$$\begin{aligned}
\boldsymbol{\dot{V}} &= \boldsymbol{\dot{e}^T M\ddot{e} + \tfrac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e}} \\
&= \boldsymbol{-\dot{e}^T C\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e - \dot{e}^T K_d \dot{e}} \\
&\quad \boldsymbol{+ \tfrac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e}} \\
&= \boldsymbol{\tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T(K_d + D)\dot{e}} \\
&= \boldsymbol{-\dot{e}^T(K_d + D)\dot{e} \;\leq\; 0}
\end{aligned}$$

$(\boldsymbol{K_d} + \boldsymbol{D})$ is positive definite

$$\dot{\boldsymbol{V}} = -\dot{\boldsymbol{e}}^{\boldsymbol{T}}(\boldsymbol{K_d} + \boldsymbol{D})\dot{\boldsymbol{e}} = \boldsymbol{0} \;\Rightarrow\; \dot{\boldsymbol{e}} = \boldsymbol{0}$$

$$\boldsymbol{M}\ddot{\boldsymbol{e}} = -\boldsymbol{C}\,\dot{\boldsymbol{e}} - \boldsymbol{D}\,\dot{\boldsymbol{e}} - \boldsymbol{K_p}\,\boldsymbol{e} - \boldsymbol{K_d}\,\dot{\boldsymbol{e}}$$

$$\dot{\boldsymbol{e}}(\boldsymbol{t}) \equiv \boldsymbol{0} \;\Rightarrow \ddot{\boldsymbol{e}}(\boldsymbol{t}) \equiv \boldsymbol{0} \;\Rightarrow\; \boldsymbol{K_p}\boldsymbol{e}(\boldsymbol{t}) \equiv \boldsymbol{0} \;\Rightarrow\; \boldsymbol{e}(\boldsymbol{t}) \equiv \boldsymbol{0}$$

By LaSalle’s theorem the origin $(\boldsymbol{e} = \boldsymbol{0}, \dot{\boldsymbol{e}} = \boldsymbol{0})$ is globally asymptotically stable

Nonlinear Systems and Control
Lecture # 12
Converse Lyapunov Functions
&
Time Varying Systems

Converse Lyapunov Theorem–Exponential Stability

Let $x = 0$ be an exponentially stable equilibrium point for the system $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{\|x\| < r\}$. Let k, λ, and r_0 be positive constants with $r_0 < r/k$ such that

$$\|x(t)\| \le k\|x(0)\|e^{-\lambda t}, \quad \forall\, x(0) \in D_0, \forall\, t \ge 0$$

where $D_0 = \{\|x\| < r_0\}$. Then, there is a continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|$$

for all $x \in D_0$, with positive constants c_1, c_2, c_3, and c_4 Moreover, if f is continuously differentiable for all x, globally Lipschitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in R^n$

Idea of the proof: Let $\psi(t;x)$ be the solution of

$$\dot{y} = f(y), \quad y(0) = x$$

Take

$$V(x) = \int_0^\delta \psi^T(t;x)\,\psi(t;x)\,dt, \quad \delta > 0$$

Example: Consider the system $\dot{x} = f(x)$ where f is continuously differentiable in the neighborhood of the origin and $f(0) = 0$. Show that the origin is exponentially stable only if $A = [\partial f / \partial x](0)$ is Hurwitz

$$f(x) = Ax + G(x)x, \quad G(x) \to 0 \text{ as } x \to 0$$

Given any $L > 0$, there is $r_1 > 0$ such that

$$\|G(x)\| \le L, \;\; \forall \; \|x\| < r_1$$

Because the origin of $\dot{x} = f(x)$ is exponentially stable, let $V(x)$ be the function provided by the converse Lyapunov theorem over the domain $\{\|x\| < r_0\}$. Use $V(x)$ as a Lyapunov function candidate for $\dot{x} = Ax$

$$\begin{aligned}\frac{\partial V}{\partial x}Ax &= \frac{\partial V}{\partial x}f(x) - \frac{\partial V}{\partial x}G(x)x \\ &\le -c_3\|x\|^2 + c_4 L\|x\|^2 \\ &= -(c_3 - c_4 L)\|x\|^2\end{aligned}$$

Take $L < c_3/c_4, \;\; \gamma \stackrel{\text{def}}{=} (c_3 - c_4 L) > 0 \;\Rightarrow$

$$\frac{\partial V}{\partial x}Ax \le -\gamma\|x\|^2, \;\; \forall \; \|x\| < \min\{r_0, \; r_1\}$$

The origin of $\dot{x} = Ax$ is exponentially stable

Converse Lyapunov Theorem–Asymptotic Stability

Let $x = 0$ be an asymptotically stable equilibrium point for $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset R^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \rightarrow \infty \text{ as } x \rightarrow \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall\, x \in R_A$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A
When $R_A = R^n$, $V(x)$ is radially unbounded

Time-varying Systems

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D$. The origin is an equilibrium point at $t = 0$ if

$$f(t, 0) = 0, \ \ \forall \, t \geq 0$$

While the solution of the autonomous system

$$\dot{x} = f(x), \quad x(t_0) = x_0$$

depends only on $(t - t_0)$, the solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

may depend on both t and t_0

Comparison Functions

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if it defined for all $r \geq 0$ and $\alpha(r) \longrightarrow \infty$ as $r \longrightarrow \infty$

- A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \longrightarrow 0$ as $s \longrightarrow \infty$

Example

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1+r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$
- $\alpha(r) = r^c$, for any positive real number c, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$
- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$

- $\beta(r,s) = r/(ksr+1)$, for any positive real number k, is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$$

Moreover, $\beta(r,s) \to 0$ as $s \to \infty$. Therefore, it belongs to class $\mathcal{KL}$

- $\beta(r,s) = r^c e^{-s}$, for any positive real number c, belongs to class $\mathcal{KL}$

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \alpha(\|x(t_0)\|), \ \forall\, t \ge t_0 \ge 0, \ \forall\, \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \ \forall\, t \ge t_0 \ge 0, \ \forall\, \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

- exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \ \forall \ \|x(t_0)\| < c$$

- globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

Theorem: Let the origin $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$ and $D \subset R^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

(1) $$W_1(x) \leq V(t, x) \leq W_2(x)$$

(2) $$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, the origin is uniformly stable

Theorem: Suppose the assumptions of the previous theorem are satisfied with

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x)$$

for all $t \ge 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \le r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r \mid W_2(x) \le c\}$ satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \ \ \forall \, t \ge t_0 \ge 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable

Theorem: Suppose the assumptions of the previous theorem are satisfied with

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a$$

for all $t \geq 0$ and $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable.

Example:

$$\dot{x} = -[1 + g(t)]x^3, \quad g(t) \geq 0, \ \forall \ t \geq 0$$

$$V(x) = \tfrac{1}{2}x^2$$

$$\dot{V}(t,x) = -[1 + g(t)]x^4 \leq -x^4, \ \ \forall \, x \in R, \, \forall \, t \geq 0$$

The origin is globally uniformly asymptotically stable

Example:

$$\begin{aligned} \dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

$$0 \leq g(t) \leq k \ \text{ and } \ \dot{g}(t) \leq g(t), \ \ \forall \, t \geq 0$$

$$V(t,x) = x_1^2 + [1+g(t)]x_2^2$$

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2, \quad \forall\, x \in R^2$$

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

$$2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$$

$$\dot{V}(t,x) \le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\, x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x$$

The origin is globally exponentially stable

Nonlinear Systems and Control
Lecture # 13
Perturbed Systems

Nominal System:

$$\dot{x} = f(x), \qquad f(0) = 0$$

Perturbed System:

$$\dot{x} = f(x) + g(t, x), \qquad g(t, 0) = 0$$

Case 1: The origin of the nominal system is exponentially stable

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \le -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4\|x\|$$

Use $V(x)$ as a Lyapunov function candidate for the perturbed system

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x)$$

Assume that

$$\|g(t,x)\| \le \gamma \|x\|, \quad \gamma \ge 0$$

$$\begin{aligned} \dot{V}(t,x) &\le -c_3\|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t,x)\| \\ &\le -c_3\|x\|^2 + c_4 \gamma \|x\|^2 \end{aligned}$$

$$\gamma < \frac{c_3}{c_4}$$

$$\dot{V}(t, x) \leq -(c_3 - \gamma c_4)\|x\|^2$$

The origin is an exponentially stable equilibrium point of the perturbed system

Example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3, \qquad \beta \geq 0 \end{aligned}$$

$$\dot{x} = Ax + g(x)$$

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}$$

The eigenvalues of A are $-1 \pm j\sqrt{3}$

$$PA + A^T P = -I \quad \Rightarrow \quad P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$$

$$V(x) = x^T P x, \quad \frac{\partial V}{\partial x} A x = -x^T x$$

$$c_3 = 1, \;\; c_4 = 2\,\|P\| = 2\lambda_{\max}(P) = 2 \times 1.513 = 3.026$$

$$\|g(x)\| = \beta |x_2|^3$$

$g(x)$ satisfies the bound $\|g(x)\| \leq \gamma \|x\|$ over compact sets of x. Consider the compact set

$$\Omega_c = \{V(x) \leq c\} = \{x^T P x \leq c\}, \quad c > 0$$

$$k_2 = \max_{x^T P x \leq c} |x_2| = \max_{x^T P x \leq c} |[0 \;\; 1] x|$$

Fact:

$$\max_{x^T P x \le c} \|Lx\| = \sqrt{c}\, \|LP^{-1/2}\|$$

Proof

$$x^T P x \le c \;\Leftrightarrow\; \frac{1}{c} x^T P x \le 1 \;\Leftrightarrow\; \frac{1}{c}\, x^T P^{1/2}\, P^{1/2} x \le 1$$

$$y = \frac{1}{\sqrt{c}}\, P^{1/2} x$$

$$\max_{x^T P x \le c} \|Lx\| = \max_{y^T y \le 1} \|L\sqrt{c}\, P^{-1/2} y\| = \sqrt{c}\, \|LP^{-1/2}\|$$

$$k_2 = \max_{x^T P x \le c} |[0 \;\; 1]x| = \sqrt{c}\, \|[0 \;\; 1]P^{-1/2}\| = 1.8194\sqrt{c}$$

$$\|g(x)\| \le \beta\, c\, (1.8194)^2 \|x\|, \quad \forall\, x \in \Omega_c$$

$$\|g(x)\| \le \gamma \|x\|, \quad \forall\, x \in \Omega_c, \quad \gamma = \beta\, c\, (1.8194)^2$$

$$\gamma < \frac{c_3}{c_4} \;\; \Leftrightarrow \;\; \beta < \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c}$$

$$\beta < 0.1/c \;\; \Rightarrow \;\; \dot{V}(x) \le -(1 - 10\beta c)\|x\|^2$$

Hence, the origin is exponentially stable and Ω_c is an estimate of the region of attraction

Alternative Bound on β

$$\begin{aligned}\dot{V}(x) &= -\|x\|^2 + 2x^T P g(x)\\ &\le -\|x\|^2 + \tfrac{1}{8}\beta x_2^3\left([2\ \ 5]x\right)\\ &\le -\|x\|^2 + \tfrac{\sqrt{29}}{8}\beta x_2^2\|x\|^2\end{aligned}$$

Over Ω_c, $x_2^2 \le (1.8194)^2 c$

$$\begin{aligned}\dot{V}(x) &\le -\left(1 - \tfrac{\sqrt{29}}{8}\beta(1.8194)^2 c\right)\|x\|^2\\ &= -\left(1 - \frac{\beta c}{0.448}\right)\|x\|^2\end{aligned}$$

If $\beta < 0.448/c$, the origin will be exponentially stable and Ω_c will be an estimate of the region of attraction

Remark: The inequality $\beta < 0.448/c$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on β

Case 2: The origin of the nominal system is asymptotically stable

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x) \le -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t,x) \right\|$$

Under what condition will the following inequality hold?

$$\left\| \frac{\partial V}{\partial x} g(t,x) \right\| < W_3(x)$$

Special Case: Quadratic-Type Lyapunov function

$$\frac{\partial V}{\partial x} f(x) \le -c_3 \phi^2(x), \qquad \left\| \frac{\partial V}{\partial x} \right\| \le c_4 \phi(x)$$

$$\dot{V}(t,x) \le -c_3\phi^2(x) + c_4\phi(x)\|g(t,x)\|$$

If $\|g(t,x)\| \le \gamma\phi(x), \quad \text{with } \gamma < \dfrac{c_3}{c_4}$

$$\dot{V}(t,x) \le -(c_3 - c_4\gamma)\phi^2(x)$$

Example

$$\dot{x} = -x^3 + g(t, x)$$

$V(x) = x^4$ is a quadratic-type Lyapunov function for the nominal system $\dot{x} = -x^3$

$$\frac{\partial V}{\partial x}(-x^3) = -4x^6, \quad \left|\frac{\partial V}{\partial x}\right| = 4|x|^3$$

$$\phi(x) = |x|^3, \quad c_3 = 4, \quad c_4 = 4$$

Suppose $|g(t, x)| \leq \gamma |x|^3, \ \forall \ x,$ with $\gamma < 1$

$$\dot{V}(t, x) \leq -4(1 - \gamma)\phi^2(x)$$

Hence, the origin is a globally uniformly asymptotically stable

Remark: A nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds

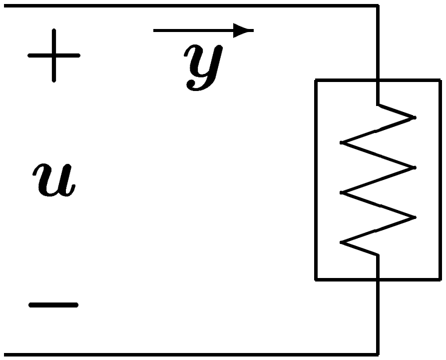
Example

$$\dot{x} = -x^3 + \gamma x$$

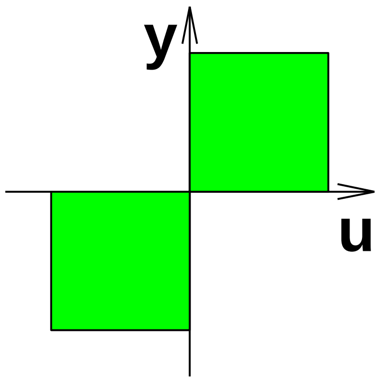
The origin is unstable for any $\gamma > 0$

Nonlinear Systems and Control
Lecture # 14
Passivity
Memoryless Functions
&
State Models

Memoryless Functions



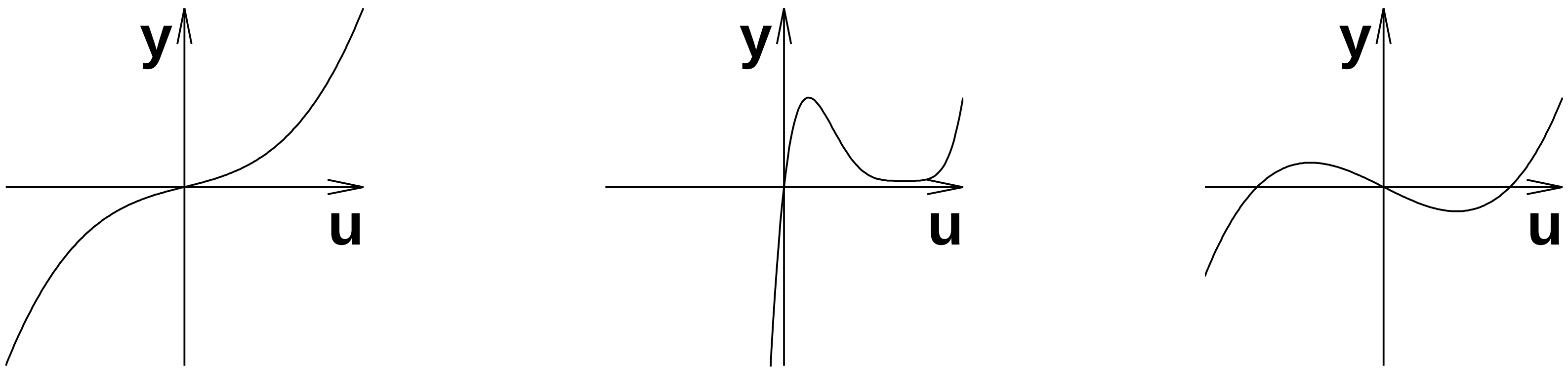
(a)



(b)

$$\text{power inflow} = uy$$

Resistor is passive if $uy \geq 0$



(a) Passive

(b) Passive

(c) Not passive

$$y = h(t, u), \quad h \in [0, \infty]$$

Vector case:

$$y = h(t, u), \quad h^T = \left[\; h_1, \;\; h_2, \;\; \cdots, \;\; h_p \; \right]$$

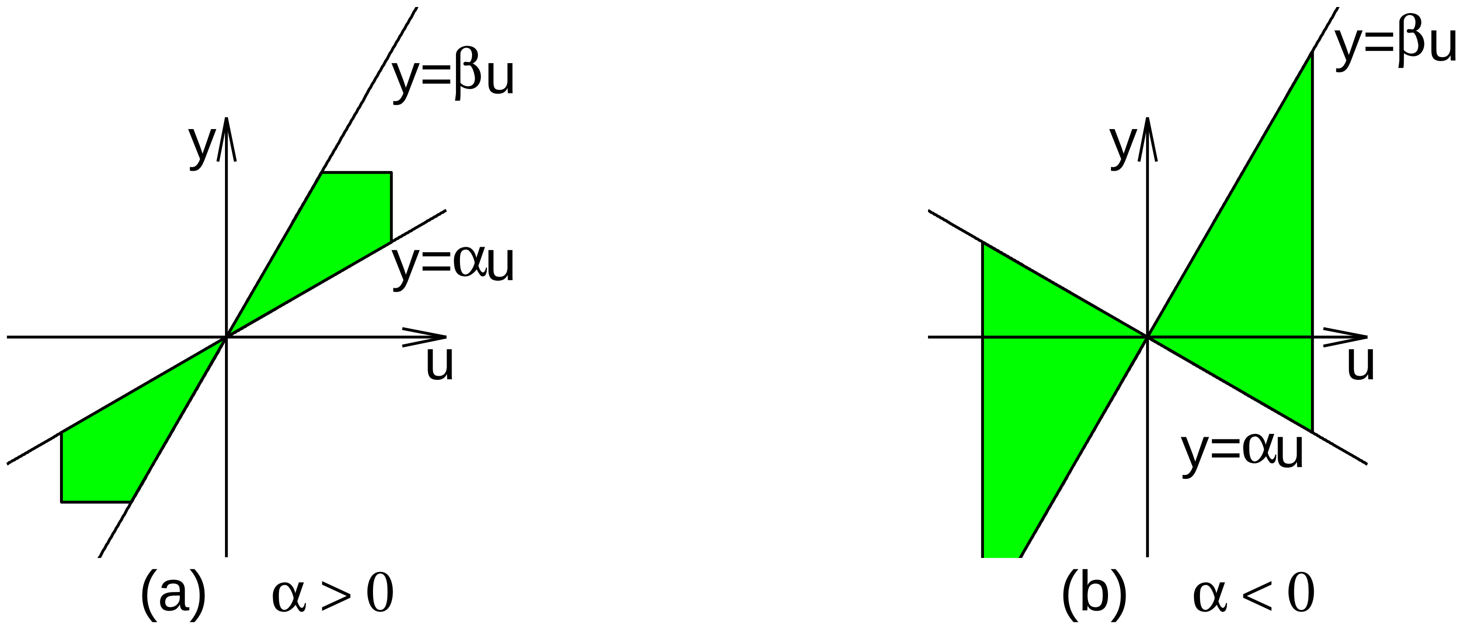
power inflow $= \Sigma_{i=1}^{p} u_i y_i = \; u^T y$

Definition: $y = h(t, u)$ is

- passive if $u^T y \geq 0$
- lossless if $u^T y = 0$
- input strictly passive if $u^T y \geq u^T \varphi(u)$ for some function φ where $u^T \varphi(u) > 0$, $\forall\, u \neq 0$
- output strictly passive if $u^T y \geq y^T \rho(y)$ for some function ρ where $y^T \rho(y) > 0$, $\forall\, y \neq 0$

Sector Nonlinearity: h belongs to the sector $[\alpha, \beta]$ ($h \in [\alpha, \beta]$) if

$$\alpha u^2 \leq u h(t, u) \leq \beta u^2$$



Also, $h \in (\alpha, \beta]$, $h \in [\alpha, \beta)$, $h \in (\alpha, \beta)$

$$\alpha u^2 \leq u h(t,u) \leq \beta u^2 \Leftrightarrow [h(t,u) - \alpha u][h(t,u) - \beta u] \leq 0$$

Definition: A memoryless function $h(t,u)$ is said to belong to the sector

- $[0, \infty]$ if $u^T h(t,u) \geq 0$
- $[K_1, \infty]$ if $u^T [h(t,u) - K_1 u] \geq 0$
- $[0, K_2]$ with $K_2 = K_2^T > 0$ if $h^T(t,u)[h(t,u) - K_2 u] \leq 0$
- $[K_1, K_2]$ with $K = K_2 - K_1 = K^T > 0$ if

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \leq 0$$

Example

$$\boldsymbol{h}(\boldsymbol{u}) = \begin{bmatrix} h_1(u_1) \\ h_2(u_2) \end{bmatrix}, \quad h_i \in [\alpha_i, \beta_i], \;\; \beta_i > \alpha_i \;\; i = 1, 2$$

$$K_1 = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \quad K_2 = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}$$

$$h \in [K_1, K_2]$$

$$K = K_2 - K_1 = \begin{bmatrix} \beta_1 - \alpha_1 & 0 \\ 0 & \beta_2 - \alpha_2 \end{bmatrix}$$

Example

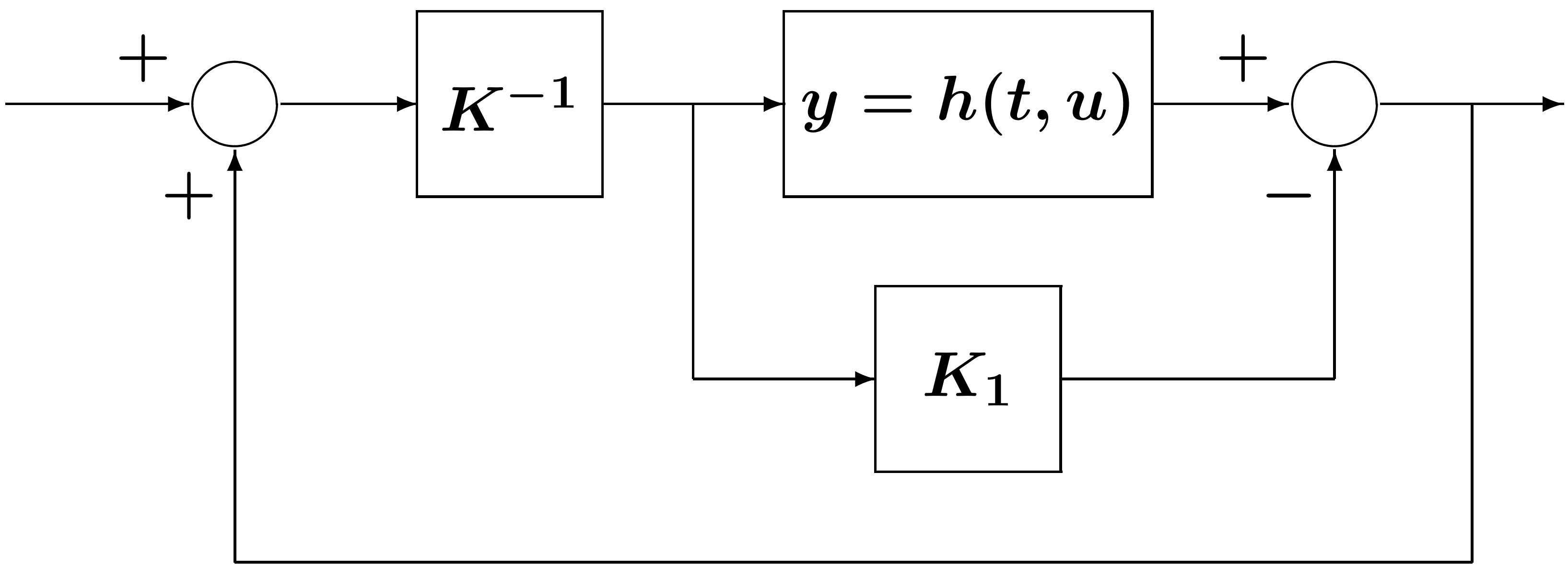
$$\|h(u) - Lu\| \leq \gamma \|u\|$$

$$K_1 = L - \gamma I, \quad K_2 = L + \gamma I$$

$$[h(u) - K_1 u]^T [h(u) - K_2 u] =$$
$$\|h(u) - Lu\|^2 - \gamma^2 \|u\|^2 \leq 0$$

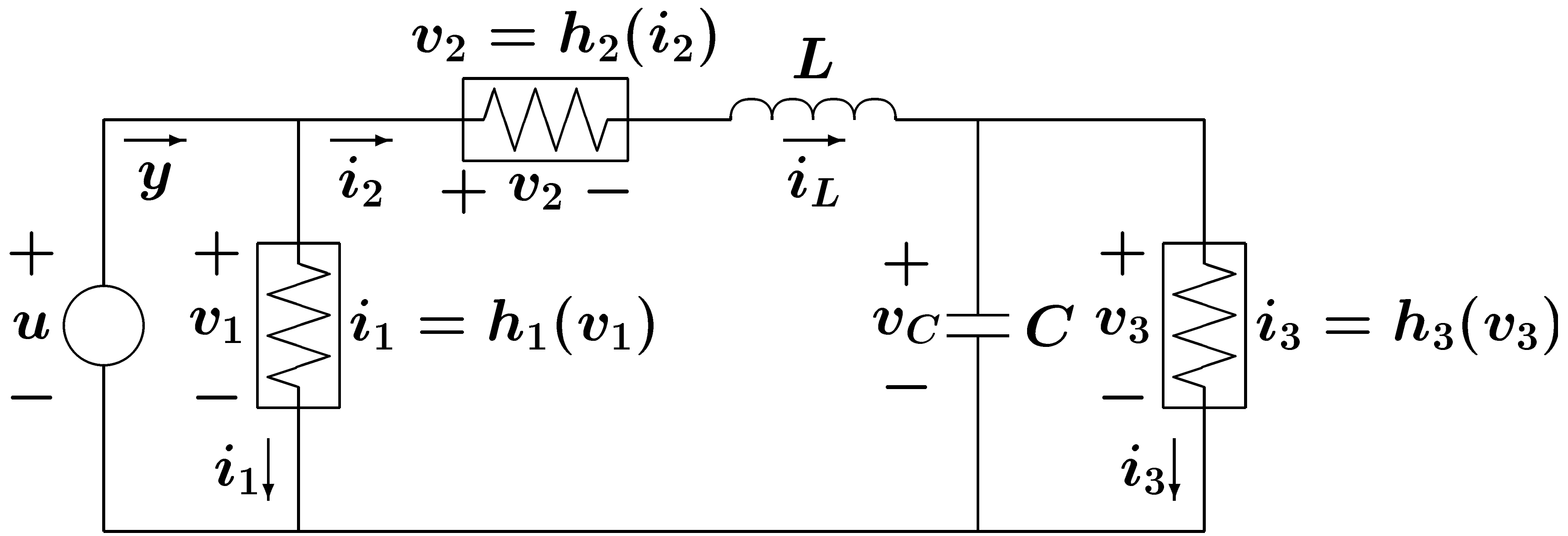
$$K = K_2 - K_1 = 2\gamma I$$

A function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback



$$[K_1, K_2] \xrightarrow{\text{Feedforward}} [0, K] \xrightarrow{K^{-1}} [0, I] \xrightarrow{\text{Feedback}} [0, \infty]$$

State Models



$$
\begin{aligned}
L\dot{x}_1 &= u - h_2(x_1) - x_2 \\
C\dot{x}_2 &= x_1 - h_3(x_2) \\
y &= x_1 + h_1(u)
\end{aligned}
$$

$$V(x) = \tfrac{1}{2}Lx_1^2 + \tfrac{1}{2}Cx_2^2$$

$$\int_0^t u(s)y(s)\,ds \geq V(x(t)) - V(x(0))$$

$$u(t)y(t) \geq \dot{V}(x(t), u(t))$$

$$\begin{aligned}
\dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 \\
&= x_1[u - h_2(x_1) - x_2] + x_2[x_1 - h_3(x_2)] \\
&= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\
&= [x_1 + h_1(u)]u - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\
&= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2)
\end{aligned}$$

$$uy = \dot{V} + uh_1(u) + x_1 h_2(x_1) + x_2 h_3(x_2)$$

If h_1, h_2, and h_3 are passive, $uy \geq \dot{V}$ and the system is passive

Case 1: If $h_1 = h_2 = h_3 = 0$, then $uy = \dot{V}$; no energy dissipation; the system is lossless

Case 2: If $h_1 \in (0, \infty]$ ($uh_1(u) > 0$ for $u \neq 0$), then

$$uy \geq \dot{V} + uh_1(u)$$

The energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the input $u(t)$ is identically zero. This is a case of input strict passivity

Case 3: If $h_1 = 0$ and $h_2 \in (0, \infty]$, then

$$y = x_1 \text{ and } uy \geq \dot{V} + yh_2(y)$$

The energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the output y is identically zero. This is a case of output strict passivity

Case 4: If $h_2 \in (0, \infty)$ and $h_3 \in (0, \infty)$, then

$$uy \geq \dot{V} + x_1 h_2(x_1) + x_2 h_3(x_2)$$

$x_1 h_2(x_1) + x_2 h_3(x_2)$ is a positive definite function of x. This is a case of state strict passivity because the energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the state x is identically zero

Definition: The system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is passive if there is a continuously differentiable positive semidefinite function $V(x)$ (the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \ \ \forall \ (x, u)$$

Moreover, it is said to be

- lossless if $u^T y = \dot{V}$
- input strictly passive if $u^T y \geq \dot{V} + u^T \varphi(u)$ for some function φ such that $u^T \varphi(u) > 0, \ \forall \ u \neq 0$

- output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ such that $y^T \rho(y) > 0, \ \forall \ y \neq 0$
- strictly passive if $u^T y \geq \dot{V} + \psi(x)$ for some positive definite function ψ

Example

$$\dot{x} = u, \qquad y = x$$

$$V(x) = \frac{1}{2}x^2 \ \Rightarrow \ uy = \dot{V} \ \Rightarrow \ \text{Lossless}$$

Example

$$\dot{x} = u, \qquad y = x + h(u), \quad h \in [0, \infty]$$

$$V(x) = \tfrac{1}{2}x^2 \;\Rightarrow\; uy = \dot{V} + uh(u) \;\Rightarrow\; \text{Passive}$$

$$h \in (0, \infty] \;\Rightarrow\; uh(u) > 0 \;\forall\; u \neq 0$$

$$\Rightarrow \; \text{Input strictly passive}$$

Example

$$\dot{x} = -h(x) + u, \qquad y = x, \quad h \in [0, \infty]$$

$$V(x) = \tfrac{1}{2}x^2 \;\Rightarrow\; uy = \dot{V} + yh(y) \;\Rightarrow\; \text{Passive}$$

$$h \in (0, \infty] \;\Rightarrow\; \text{Output strictly passive}$$

Example

$$\dot{x} = u, \qquad y = h(x), \quad h \in [0, \infty]$$

$$V(x) = \int_0^x h(\sigma)\, d\sigma \;\Rightarrow\; \dot{V} = h(x)\dot{x} = yu \;\Rightarrow\; \text{Lossless}$$

Example

$$a\dot{x} = -x + u, \qquad y = h(x), \quad h \in [0, \infty]$$

$$V(x) = a \int_0^x h(\sigma)\, d\sigma \;\Rightarrow\; \dot{V} = h(x)(-x+u) = yu - xh(x)$$

$$yu = \dot{V} + xh(x) \;\Rightarrow\; \text{Passive}$$

$$h \in (0, \infty] \;\Rightarrow\; \text{Strictly passive}$$

Nonlinear Systems and Control
Lecture # 15
Positive Real Transfer Functions
&
Connection with Lyapunov Stability

Definition: A $p \times p$ proper rational transfer function matrix $G(s)$ is positive real if

- poles of all elements of $G(s)$ are in $Re[s] \leq 0$
- for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is called strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$

Scalar Case ($p = 1$):

$$G(j\omega) + G^T(-j\omega) = 2Re[G(j\omega)]$$

$Re[G(j\omega)]$ is an even function of ω. The second condition of the definition reduces to

$$Re[G(j\omega)] \geq 0, \ \forall \ \omega \in [0, \infty)$$

which holds when the Nyquist plot of of $G(j\omega)$ lies in the closed right-half complex plane

This is true only if the relative degree of the transfer function is zero or one

Lemma: A $p \times p$ proper rational transfer function matrix $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega) + G^T(-j\omega) > 0, \; \forall\, \omega \in R$
- $G(\infty) + G^T(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^{2(p-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$$

where $q = \text{rank}[G(\infty) + G^T(\infty)]$

Scalar Case ($p = 1$): $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $Re[G(j\omega)] > 0, \ \forall\, \omega \in [0, \infty)$
- $G(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] > 0$$

Example:

$$G(s) = \frac{1}{s}$$

has a simple pole at $s = 0$ whose residue is 1

$$Re[G(j\omega)] = Re\left[\frac{1}{j\omega}\right] = 0, \quad \forall\, \omega \neq 0$$

Hence, G is positive real. It is not strictly positive real since

$$\frac{1}{(s - \varepsilon)}$$

has a pole in $Re[s] > 0$ for any $\varepsilon > 0$

Example:

$$G(s) = \frac{1}{s+a}, \;\; a > 0, \quad \text{is Hurwitz}$$

$$Re[G(j\omega)] = \frac{a}{\omega^2 + a^2} > 0, \;\; \forall\, \omega \in [0, \infty)$$

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] = \lim_{\omega \to \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0 \;\; \Rightarrow \quad \text{G is SPR}$$

Example:

$$G(s) = \frac{1}{s^2 + s + 1}, \quad Re[G(j\omega)] = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2}$$

G is not PR

Example:

$$G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix} \quad \text{is Hurwitz}$$

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} > 0, \quad \forall\, \omega \in R$$

$$G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = 1$$

$$\lim_{\omega \to \infty} \omega^2 \det[G(j\omega) + G^T(-j\omega)] = 4 \quad \Rightarrow \quad G \text{ is SPR}$$

Positive Real Lemma: Let

$$G(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (A, C) is observable. $G(s)$ is positive real if and only if there exist matrices $P = P^T > 0$, L, and W such that

$$\begin{aligned} PA + A^TP &= -L^TL \\ PB &= C^T - L^TW \\ W^TW &= D + D^T \end{aligned}$$

Kalman–Yakubovich–Popov Lemma: Let

$$G(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (A, C) is observable. $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, L, and W, and a positive constant ε such that

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

Lemma: The linear time-invariant minimal realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with

$$G(s) = C(sI - A)^{-1}B + D$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Proof: Apply the PR and KYP Lemmas, respectively, and use $V(x) = \frac{1}{2}x^T Px$ as the storage function

$$
\begin{aligned}
& u^T y - \frac{\partial V}{\partial x}(Ax + Bu) \\
&\quad = \; u^T(Cx + Du) - x^T P(Ax + Bu) \\
&\quad = \; u^T Cx + \tfrac{1}{2}u^T(D + D^T)u \\
&\qquad - \tfrac{1}{2}x^T(PA + A^T P)x - x^T PBu \\
&\quad = \; u^T(B^T P + W^T L)x + \tfrac{1}{2}u^T W^T W u \\
&\qquad + \tfrac{1}{2}x^T L^T Lx + \tfrac{1}{2}\varepsilon x^T Px - x^T PBu \\
&\quad = \; \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) + \tfrac{1}{2}\varepsilon x^T Px \; \geq \tfrac{1}{2}\varepsilon x^T Px
\end{aligned}
$$

In the case of the PR Lemma, $\varepsilon = 0$, and we conclude that the system is passive; in the case of the KYP Lemma, $\varepsilon > 0$, and we conclude that the system is strictly passive

Connection with Lyapunov Stability

Lemma: If the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is passive with a positive definite storage function $V(x)$, then the origin of $\dot{x} = f(x,0)$ is stable

Proof:

$$u^T y \geq \frac{\partial V}{\partial x} f(x,u) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x,0) \leq 0$$

Lemma: If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is strictly passive, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof: The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \psi(x) \;\Rightarrow\; \frac{\partial V}{\partial x} f(x, 0) \leq -\psi(x)$$

Why is $V(x)$ positive definite? Let $\phi(t; x)$ be the solution of $\dot{z} = f(z, 0)$, $z(0) = x$

$$\dot{V} \leq -\psi(x)$$

$$V(\phi(\tau, x)) - V(x) \leq -\int_0^{\tau} \psi(\phi(t; x))\, dt, \quad \forall\, \tau \in [0, \delta]$$

$$V(\phi(\tau, x)) \geq 0 \;\; \Rightarrow \;\; V(x) \geq \int_0^{\tau} \psi(\phi(t; x))\, dt$$

$$V(\bar{x}) = 0 \;\Rightarrow\; \int_0^{\tau} \psi(\phi(t; \bar{x}))\, dt = 0,\; \forall\, \tau \in [0, \delta]$$

$$\Rightarrow\; \psi(\phi(t; \bar{x})) \equiv 0 \;\Rightarrow\; \phi(t; \bar{x}) \equiv 0 \;\Rightarrow\; \bar{x} = 0$$

Definition: The system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \{h(x, 0) = 0\}$, other than the zero solution $x(t) \equiv 0$

Linear Systems

$$\dot{x} = Ax, \quad y = Cx$$

Observability of (A, C) is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \;\Leftrightarrow\; x(0) = 0 \;\Leftrightarrow\; x(t) \equiv 0$$

Lemma: If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is output strictly passive and zero-state observable, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof: The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + y^T \rho(y) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x, 0) \leq -y^T \rho(y)$$

$$\dot{V}(x(t)) \equiv 0 \; \Rightarrow \; y(t) \equiv 0 \; \Rightarrow x(t) \equiv 0$$

Apply the invariance principle

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0$$

$$V(x) = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2$$

$$\dot{V} = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

The system is output strictly passive

$$y(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

The system is zero-state observable. V is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable