Elementary approach to the derivation of Kepler's laws

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Abstract

In this short text, we will introduce, based on the calculus and the laws of motion in classical mechanics, a strong mathematical derivation of Kepler's famous astronomical laws.

We refer by M to the Sun's mass, and let us imagine that there is an inertial physical observer, using a clock to measure time and an orthonormal basis to determine the spatial locations of the various celestial bodies from the center of the sun. The presence of inertial observers is one of the fundamental postulates of classical mechanics.

For this observer, the path of motion of a planet with a mass m around the sun is represented by a parameterized curve $\gamma : [0, T[\rightarrow \mathbb{R}^3$, which verifies the following differential equation called **the Newton's equation**:

$$m\ddot{\gamma}(t) = F(t, \gamma(t), \dot{\gamma}(t)), \tag{1}$$

where F indicates the sum of the forces acting on the planet. We will assume here that the force field F is centripetal; that is, F hase the mathematical formula

$$F(\mathbf{r}) = -f(\mathbf{r})\,\mathbf{r}, \mathbf{r} = (x, y, z) \in \mathbb{R}^3 - \{(0,0,0)\},$$

with $f \in C^\infty(\mathbb{R}^3 - \{(0,0,0)\}; \mathbb{R})$.

The angular momentum of the planet is

$$J(t) = \gamma(t) \times \dot{\gamma}(t), t \in [0, T[.$$

Lemma 0.1 *The angular momentum J is constant.*

Proof. Indeed, for each $t \in [0, T[$, we have

$$\begin{aligned} \dot{J}(t) &= \dot{\gamma}(t) \times \dot{\gamma}(t) + \gamma(t) \times \ddot{\gamma}(t) = 0 + \gamma(t) \times (-f(\gamma(t))\gamma(t)) \\ &= -f(\gamma(t))\gamma(t) \times \gamma(t) = 0. \end{aligned}$$

So J is constant in $[0, T[$. ■

Based on this lemma and the properties of the vector product in $\mathbb{R}^3$, we conclude that the planet moves in the plane passing through the origin $(0,0,0) \in \mathbb{R}^3$ (which physically represents the center of the sun) and orthogonal to the angular momentum vector J. So, we get the important physical reality: **the movement of the planet is a flat movement**.

On the other hand, we can choose new coordinates of space $\mathbb{R}^3$ so that the Cartesian equation of the plane at which the planet moves becomes the equation $z = 0$ and that the angular momentum vector J becomes as follows $J = (0, 0, j)$ with $j \in \mathbb{R}$. In this case, for $t \in [0, T[$, we write

$$\gamma(t) = (x(t), y(t), 0) = (r(t)\cos\theta(t), r(t)\sin\theta(t), 0),$$

where $r(t) = \|\gamma(t)\|_2 = \sqrt{(x(t))^2 + (y(t))^2}$ is the Euclidean norm of the vector $\gamma(t)$. The derivative $\dot{\theta}(t)$ is called the angular velocity of the planet at the moment t.

For $t \in [0, T[$, we put $Z(t) = x(t) + iy(t) = r(t)\exp(i\theta(t)) \in \mathbb{C}$, and we denote by $\operatorname{Im}(Z)$ the imaginary part of the complex number $Z \in \mathbb{C}$.

Lemma 0.2 *For each* $t \in [0, T[$, $j = (r(t))^2\,\dot{\theta}(t) \geq 0$.

Proof. Let $t \in [0, T[$. We have

$$\begin{aligned} J(t) &= (0,0,j) = \gamma(t) \times \dot{\gamma}(t) = (x(t), y(t), 0) \times (\dot{x}(t), \dot{y}(t), 0) \\ &= (0, 0, x(t)\dot{y}(t) - \dot{x}(t)y(t)). \end{aligned}$$

Then

$$\begin{aligned} j &= x(t)\dot{y}(t) - \dot{x}(t)y(t) = \operatorname{Im}\left(\overline{Z(t)}\dot{Z}(t)\right) \\ &= \operatorname{Im}\left[r(t)\exp(-i\theta(t))\left\{\dot{r}(t)\exp(i\theta(t)) + ir(t)\dot{\theta}(t)\exp(i\theta(t))\right\}\right] \\ &= \operatorname{Im}\left[r(t)\dot{r}(t) + i(r(t))^2\dot{\theta}(t)\right], \end{aligned}$$

and thus $j = (r(t))^2\,\dot{\theta}(t)$. ■

Hence, the quantity $j = (r(t))^2\,\dot{\theta}(t)$ is constant and does not change with time.

The flat region swept by the vector extending from the center of the sun to the center of the planet between moments t_1 and t_2 can be mathematically represented as follows:

$$\mathcal{R}(t_1,t_2) = \{(x,y,0) = (r\cos\theta(t), r\sin\theta(t), 0) : t_1 \leq t \leq t_2, 0 \leq r \leq r(t)\}.$$

Let's denote by $\operatorname{Area}(t_1,t_2)$ to the area of this region.

Lemma 0.3 *For all* $(t_1,t_2) \in \mathbb{R}^2$ *with* $0 \leq t_1 \leq t_2 \leq T$*, we have*

$$\operatorname{Area}(t_1,t_2) = \frac{j}{2}(t_2 - t_1).$$

Proof. Let $(t_1,t_2) \in \mathbb{R}^2$ with $0 \leq t_1 \leq t_2 \leq T$. According to the integration and the measure theory, this area is calculated as follows:

$$\begin{aligned}
\operatorname{Area}(t_1,t_2) &= \int_{\mathcal{R}(t_1,t_2)} dxdy = \int_{t_1\leq t\leq t_2, 0\leq r\leq r(t)} \left|\det\begin{pmatrix} \partial_t x & \partial_r x \\ \partial_t y & \partial_r y \end{pmatrix}\right| dtdr \\
&= \int_{t_1\leq t\leq t_2, 0\leq r\leq r(t)} \left|\det\begin{pmatrix} -r\dot{\theta}(t)\sin\theta(t) & \cos\theta(t) \\ r\dot{\theta}(t)\cos\theta(t) & \sin\theta(t) \end{pmatrix}\right| dtdr \\
&= \int_{t_1}^{t_2} \dot{\theta}(t)\left(\int_0^{r(t)} rdr\right) dt = \frac{1}{2}\int_{t_1}^{t_2} (r(t))^2\,\dot{\theta}(t)\,dt \\
&= \frac{1}{2}\int_{t_1}^{t_2} jdt = \frac{j}{2}\int_{t_1}^{t_2} dt = \frac{j}{2}(t_2 - t_1).
\end{aligned}$$

This is the relationship to be reached. ■

From Lemma 0.3, we extract an ancient and very famous astronomical law called **the Kepler's second law: the flat region $\mathcal{R}(t_1,t_2)$ is swept out at a constant rate $j\,|2$ by the vector extending from the center of the sun to the center of the planet.**

Kepler wrote this law in 1602 and published it in 1606. In fact, Kepler formulated three famous astronomical laws, which were the result of direct observations to the sky for very long periods extending to an ancient history. In a later period, Newton could verify, on the paper only, the validity of these laws, using the calculus, laws of motion in classical mechanics and the general law of gravitation. This incredible scientific success that Newton achieved that made him very famous in the world, just as it made him one of the greatest scientists in earth.

Now we specify the force field F. According to **the Newton's law of gravity** the sun acts on a planet of mass m at the point $\mathbf{r} = (x,y,z) \in \mathbb{R}^3 - \{(0,0,0)\}$ by the force

$$F(\mathbf{r}) = -GMm\frac{\mathbf{r}}{r^3} = -\operatorname{grad} U(\mathbf{r}) = -(\partial_x U(\mathbf{r}), \partial_y U(\mathbf{r}), \partial_z U(\mathbf{r})),$$

where $G = (6.67428 \pm 0.00067) \times 10^{-11}$ is the gravitational constant, $r = \|\mathbf{r}\|_2 = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean norm of the vector $\mathbf{r}$ and U is the gravitational potential defined by

$$U(\mathbf{r}) = -G\frac{Mm}{r}.$$

In this case, the equation of motion of the planet (1) becomes in the following form

$$\ddot{\gamma}(t) = -GM\frac{\gamma(t)}{(r(t))^3}. \tag{2}$$

From here, we can easily see that **the motion of the planet around the sun depends only on the initial position** $\gamma(0)$ **and the initial velocity** $\dot{\gamma}(0)$**, and it does not depend on its mass** m.

The three energies of the planet are the kinetic energy E_{k}, the potential energy E_{p} and the total energy E, which are defined as follows

$$E_{\mathrm{k}}(t) = \frac{1}{2}m\|\dot{\gamma}(t)\|_2^2, E_{\mathrm{p}}(t) = U(\gamma(t)), E(t) = E_{\mathrm{k}}(t) + E_{\mathrm{p}}(t), t \in [0, T[.$$

Lemma 0.4 *The total energy E of the planet is constant along its orbit.*