حل السلسلة رقم 02

التمرين 01

حساب معكوس مصفوفة:

$$A^{-1} = \frac{1}{|A|} \cdot adj\,A .$$

$adj\,A$: هي منقول مصفوفة المرافقات وتحسب كما يلي:

إذا كانت المصفوفة من الشكل:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ; \quad adj\,A = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

أما إذا كانت المصفوفة ثلاثية فانها تحسب كما يلي:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$adj\,a = \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} ، |A| = 2 ، adj\,A = \begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1 & 2 \end{pmatrix}$$

حتى تكون A^{-1} صحيحة لابد من تحقق العلاقة التالية:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

$$\begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & -1/2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix} ، |A| = 14 \quad adj\,A = \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 2/7 & 1/7 \\ -1/14 & 3/14 \end{pmatrix}$$

$$A^{-1} \cdot A = \begin{pmatrix} 2/7 & 1/7 \\ -1/14 & 3/14 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} ، |A| = 1$$

$$adj\,A = {}^{t}\begin{pmatrix} -11 & -4 & 6 \\ 2 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & 4 & 1 \end{pmatrix} ، |A| = -24$$

$$adj\,A = {}^{t}\begin{pmatrix} -9 & -1 & 13 \\ -6 & 2 & -2 \\ 3 & -5 & -7 \end{pmatrix} = \begin{pmatrix} -9 & -6 & 3 \\ -1 & 2 & -5 \\ 13 & -2 & -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-24}\begin{pmatrix} -9 & -6 & 3 \\ -1 & 2 & -5 \\ 13 & -2 & -7 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 3 & 2 & 1 \end{pmatrix} ، |A| = 10$$

$$adj\,A = {}^{t}\begin{pmatrix} -5 & 3 & 9 \\ 0 & -2 & 4 \\ 5 & 1 & -7 \end{pmatrix} = \begin{pmatrix} -5 & 0 & 5 \\ 3 & -2 & 1 \\ 9 & 4 & -7 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1/2 & 0 & 1/2 \\ 3/10 & -1/5 & 1/10 \\ 9/10 & 2/5 & -7/10 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} , |A| = 11$$

$$\operatorname{adj} A = {}^{t}\begin{pmatrix} -1 & -2 & 5 \\ 5 & -1 & -3 \\ -3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 5 & -3 \\ -2 & -1 & 5 \\ 5 & -3 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{pmatrix} -1 & 5 & -3 \\ -2 & -1 & 5 \\ 5 & -3 & 4 \end{pmatrix} . \checkmark$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 3 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} L_2' = L_2 + L_1 \\ L_3' = L_3 + L_1 \\ L_4' = L_4 - L_1 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) L_3' = L_3 - \tfrac{1}{4} L_2$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 2 & \frac{3}{4} & -\frac{1}{4} & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) L_4' = L_4 - 4L_3$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 4 & -2 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 2 & \frac{3}{4} & -\frac{1}{4} & 1 & 0 \\ 0 & 0 & 0 & -7 & -4 & 1 & -4 & 1 \end{array}\right)$$

بعد ما قمنا بتحويل المصفوفة الى مثلثية علوية نقوم بتحويل عناصر القطر الرئيسي الى 1 وذلك بقسمة كل العنصر على نفسه .

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 4 & \frac{3}{2} & -\frac{1}{2} & 2 & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{7} & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{7} \end{array}\right)$$

ثم نقوم بتحويل العناصر التي فوق القطر الرئيسي في المصفوفة المعطاة الى أصفار بنفس الطريقة .

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 4 & \frac{3}{2} & -\frac{1}{2} & 2 & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{7} & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{7} \end{array}\right) \begin{array}{l} L_1' = L_1 - L_4 \\ L_2' = L_2 - L_4 \\ L_3' = L_3 - 4L_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 0 & \frac{3}{7} & \frac{1}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{9}{28} & \frac{11}{28} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 1 & 0 & -\frac{11}{14} & \frac{1}{14} & -\frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 0 & 1 & \frac{4}{7} & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{7} \end{array}\right) \begin{array}{l} L_1' = L_1 + L_3 \\ L_2' = L_2 + \frac{1}{2} L_3 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & \frac{5}{14} & \frac{3}{14} & -\frac{6}{7} & \frac{5}{7} \\ 0 & 1 & 0 & 0 & -\frac{10}{14} & \frac{6}{14} & -\frac{5}{7} & \frac{3}{7} \\ 0 & 0 & 1 & 0 & -\frac{11}{14} & \frac{1}{14} & -\frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 0 & 1 & \frac{4}{7} & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{7} \end{array}\right) L_1' = L_1 - 2L_2$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{15}{14} & -\frac{9}{14} & \frac{4}{7} & -\frac{1}{7} \\ 0 & 1 & 0 & 0 & -\frac{10}{14} & \frac{6}{14} & -\frac{5}{7} & \frac{3}{7} \\ 0 & 0 & 1 & 0 & -\frac{11}{14} & \frac{1}{14} & -\frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 0 & 1 & \frac{4}{7} & -\frac{1}{7} & \frac{4}{7} & -\frac{1}{7} \end{array}\right)$$

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -15 & 9 & -8 & 2 \\ 10 & -6 & 10 & -6 \\ 11 & -1 & 4 & -8 \\ -8 & 2 & -8 & 2 \end{pmatrix}$$

$$|E| = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 1 & 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & 1 & 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & 1 & 2 & 4 & 6 & 1 \end{vmatrix}$$

السطر الأول والثالث والخامس متساويين وبالتالي $|E| = 0$

التمرين 3:

حل الأنظمة الخطية بطريقة GAUSS

$$S_1 \begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 1 \\ 3x_1 - x_2 - x_3 - 2x_4 = -4 \\ 2x_1 + 3x_2 - x_3 - x_4 = -6 \\ x_1 + 2x_2 + 3x_3 - x_4 = -4 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 1 \\ 3 & -1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 & -6 \\ 1 & 2 & 3 & -1 & -4 \end{array}\right) \begin{array}{l} L_2' = L_2 - 3L_1 \\ L_3' = L_3 - 2L_1 \\ L_4' = L_4 - L_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 1 \\ 0 & -4 & -7 & -11 & -7 \\ 0 & 1 & -5 & -7 & -8 \\ 0 & 1 & 1 & -4 & -5 \end{array}\right) \quad L_2 \longleftrightarrow L_4$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & 1 & -4 & -5 \\ 0 & 1 & -5 & -7 & -8 \\ 0 & -4 & -7 & -11 & -7 \end{array}\right) \begin{array}{l} L_3' = L_3 - L_2 \\ L_4' = L_4 + 4L_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -4 & -5 \\ 0 & 0 & -6 & -3 & -3 \\ 0 & 0 & -27 & -39 & -39 \end{array}\right) \quad L_4 = L_4 - \frac{27}{6} L_3$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & -6 & -3 & -3 \\ 0 & 0 & 0 & -\frac{51}{2} & -\frac{51}{2} \end{array}\right) \begin{array}{l} x_1 = -1 \\ x_2 = -1 \\ x_3 = 0 \\ x_4 = 1 \end{array}$$

التمرين 2:

حساب المحددات.

$$|A| = \begin{vmatrix} 2 & -1 & 4 & 1 & 0 \\ 3 & 2 & 6 & 3 & 0 \\ 1 & 3 & 2 & -1 & 2 \\ 2 & 1 & 4 & 2 & 0 \\ 1 & 2 & 2 & 1 & 1 \end{vmatrix}$$

نلاحظ أن العمود الثاني ضعف العمود الأول وبالتالي المحدد يساوي صفر ودون أن نحسب وهذا من خواص المحددات.

$$|B| = \begin{vmatrix} 1 & 2 & 1 & 0 & 4 & 3 \\ -1 & 1 & -1 & 2 & 1 & 3 \\ 2 & 1 & 3 & 5 & 2 & 1 \\ 0 & 3 & 0 & 2 & 5 & 6 \\ 1 & 2 & 3 & 2 & -1 & 2 \\ -1 & 1 & -1 & 2 & 1 & 3 \end{vmatrix}$$

السطر الثاني يساوي السطر الأخير وبالتالي المحدد يساوي الصفر.

$$|C| = \begin{vmatrix} 1 & 2 & 1 & -1 & 2 \\ 0 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 2 \end{vmatrix}$$

$$|C| = 1 \times 2 \times 1 \times \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} = 12$$

$$|D| = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1 \times 4 \times 3 \times 2 = 24$$

$$S_4 = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ 3 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -4 \\ 0 & -2 & -2 & -2 \end{array}\right) L_3' = L_3 - 2L_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -4 \\ 0 & 0 & 4 & 6 \end{array}\right) \begin{array}{l} x_1 = 0 \\ x_2 = -1/2 \\ x_3 = 3/2 \end{array}$$

حل التمرين 4 :

$$S_1 \begin{cases} x_1 + \lambda x_2 + x_3 = 2 \\ x_1 - x_2 + x_3 = -1 \\ x_1 + 2x_2 - x_3 = 1 \end{cases}$$

حتى تكون للجملة حلا وحيدا لابد أن يكون $|S_1| \neq 0$.

$$|S_1| = \begin{vmatrix} 1 & \lambda & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 2\lambda + 2$$

$$2\lambda + 2 = 0 \Rightarrow \lambda = -1$$

إذن $\forall \lambda \in \mathbb{R} - \{-1\}$ فإن للجملة حل

$$|S_2| = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} = 6$$

إذن $\forall \lambda \in \mathbb{R}$ فإن للجملة حلا و

$$|S_3| = \begin{vmatrix} 1 & 1 & 1 \\ \lambda & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 3\lambda \neq 0$$

$\forall \lambda \in \mathbb{R}^*$ فإن للجملة حلا وحيدا

$$|S_4| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ \lambda & 1 & 1 \end{vmatrix} = 2 - 2\lambda$$

$\forall \lambda \in \mathbb{R} - \{1\}$ فإن للجملة حل وحيد

$$S_2 \begin{cases} x_1 + x_2 + x_3 = 5 \\ 2x_1 + x_2 - x_3 = 1 \\ 3x_1 + 2x_2 - x_3 = 4 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & -1 & 4 \end{array}\right) \begin{array}{l} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -1 & -3 & -9 \\ 0 & -1 & -4 & -11 \end{array}\right) L_3' = L_3 - L_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -1 & -3 & -9 \\ 0 & 0 & -1 & -2 \end{array}\right) \begin{array}{l} x_1 = 0 \\ x_2 = 3 \\ x_3 = 2 \end{array}$$

$$S_3 \begin{cases} x_1 + 2x_2 + x_3 + x_4 = 1 \\ x_1 - x_2 + 2x_3 - x_4 = -1 \\ 2x_1 + x_2 + 3x_3 - x_4 = 2 \\ 2x_1 - x_2 - x_3 + x_4 = 1 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & 3 & -1 & 2 \\ 2 & -1 & -1 & 1 & 1 \end{array}\right) \begin{array}{l} L_2' = L_2 - L_1 \\ L_3' = L_3 - 2L_1 \\ L_4' = L_4 - 2L_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -2 \\ 0 & -3 & 1 & -3 & 0 \\ 0 & -5 & -3 & -1 & -1 \end{array}\right) \begin{array}{l} L_3' = L_3 - L_2 \\ L_4' = L_4 - \frac{5}{3}L_3 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -2 \\ 0 & 0 & -\frac{14}{3} & 4 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -2 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & -\frac{14}{3} & 4 & -1 \end{array}\right) L_3 \leftrightarrow L_4$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -2 \\ 0 & 0 & -\frac{14}{3} & 4 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) \begin{array}{l} x_1 = 3/2 \\ x_2 = 3/2 \\ x_3 = -3/2 \\ x_4 = -2 \end{array}$$

- حل السلسلة رقم 03 -

التمرين 1 :

$$S_1 \begin{cases} x_1 - x_2 = 3 \\ 2x_1 + x_3 = 9 \end{cases} \qquad \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

$|S_1| = 3$ للجملة حل.

$$S_2 \begin{cases} x_1 + 3x_2 = 6 \\ 2x_1 - x_2 = 5 \end{cases} \qquad \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$|S_2| = -7$ للجملة حل.

$$S_3 \begin{cases} x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 + 2x_3 = 0 \\ 2x_1 + 3x_3 = 2 \end{cases} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$|S_3| = 0$ لا يوجد للجملة حل وحيد.

$$S_4 \begin{cases} 2x_1 + x_2 - x_3 = 2 \\ x_2 + 3x_3 = 3 \\ 2x_1 + 3x_2 + 2x_3 = -5 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

$|S_4| = -6$ يوجد للجملة حل وحيد.

التمرين 2 :

$$S_1 \begin{cases} x_1 + x_2 + 2x_3 = -1 \\ 2x_1 - x_2 + 2x_3 = -4 \\ 4x_1 + x_2 + 4x_3 = -2 \end{cases}$$

1/ طريقة مقلوب مصفوفة :

$X = A^{-1} . b$

$A^{-1} = \frac{1}{|A|} . \mathrm{adj}\,A \;,\; |A| = 6$

$\mathrm{adj}\,A = {}^t\begin{pmatrix} -6 & 0 & 6 \\ -2 & -4 & 4 \\ \ldots \end{pmatrix} = \begin{pmatrix} -6 & -2 & 4 \\ 0 & -4 & 2 \\ \ldots \end{pmatrix}$

$$X = \frac{1}{6} \begin{pmatrix} -6 & -2 & 4 \\ 0 & -4 & 2 \\ 6 & 3 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

طريقة كرامر :

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} -1 & 1 & 2 \\ -4 & -1 & 2 \\ -2 & 1 & 4 \end{vmatrix}}{6} = \frac{6}{6} = 1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & 2 \\ 2 & -4 & 2 \\ 4 & -2 & 4 \end{vmatrix}}{6} = \frac{12}{6} = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & -4 \\ 4 & 1 & -2 \end{vmatrix}}{6} = \frac{-12}{6} = 2$$

طريقة غوص :

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & -1 & 2 & -4 \\ 4 & 1 & 4 & -2 \end{array}\right) \begin{array}{l} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 4L_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -2 & -2 \\ 0 & -3 & -4 & 2 \end{array}\right) L_3' = L_3 - L_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -3 & -2 & -2 \\ 0 & 0 & -2 & 4 \end{array}\right) \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = -2 \end{array}$$

$$S_2 = \begin{cases} x_1 - x_2 + x_3 = 4 \\ x_1 - 2x_2 + x_3 = 2 \\ 3x_1 + x_2 + 2x_3 = 1 \end{cases}$$

1/ طريقة المقلوب :

$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 1 \\ 3 & 1 & 2 \end{pmatrix} \;,\; |A| = 1$

$\mathrm{adj}\,A = \begin{pmatrix} -5 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & -4 & -1 \end{pmatrix}$

$$X = \frac{1}{1} \begin{pmatrix} -5 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & -4 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -13 \\ 2 \\ 19 \end{pmatrix}$$

2/ طريقة كرامر:

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 4 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 2 \end{vmatrix}}{1} = \frac{-13}{1} = -13$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}}{1} = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & 4 \\ 1 & -2 & 2 \\ 3 & 1 & 1 \end{vmatrix}}{1} = 19$$

3/ طريقة غوص:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 1 & -2 & 1 & 2 \\ 3 & 1 & 2 & 1 \end{array}\right) \begin{array}{l} L_2' = L_2 - L_1 \\ L_3' = L_3 - L_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & +4 & -1 & -11 \end{array}\right) L_3' = L_3 + 4L_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & -19 \end{array}\right) \begin{array}{l} x_1 = -13 \\ x_2 = 2 \\ x_3 = 19 \end{array}$$

$$S_3 = \begin{cases} x_1 + x_2 + 2x_3 = 5 \\ 2x_1 + x_2 + x_3 = 4 \\ 3x_1 - x_2 + 3x_3 = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & -1 & 3 \end{pmatrix} , |A| = -9$$

1/ مقلوب مصفوفة:

$$\mathrm{adj}\,A = \begin{pmatrix} 4 & -5 & -1 \\ -3 & -3 & 3 \\ -5 & 4 & -1 \end{pmatrix}$$

$$X = A^{-1} \cdot b = -\frac{1}{9} \begin{pmatrix} 4 & -5 & -1 \\ -3 & -3 & 3 \\ -5 & 4 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 21/9 \\ \cdots \end{pmatrix}$$

2/ طريقة كرامر:

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 5 & 1 & 2 \\ 4 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix}}{-9} = \frac{-2}{-9} = \frac{2}{9}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 5 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 3 \end{vmatrix}}{-9} = \frac{-21}{-9} = \frac{21}{9}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{vmatrix}}{-9} = \frac{-11}{-9} = \frac{11}{9}$$

3/ طريقة غوص:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 2 & 1 & 1 & 4 \\ 3 & -1 & 3 & 2 \end{array}\right) \begin{array}{l} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -1 & -3 & -6 \\ 0 & -4 & -3 & -13 \end{array}\right) L_3' = L_3 - 4L_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -1 & -3 & -6 \\ 0 & 0 & 9 & 11 \end{array}\right) \begin{array}{l} x_1 = 2/9 \\ x_2 = 21/9 \\ x_3 = 11/9 \end{array}$$

$$S_4 \begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 2x_1 - x_2 + 3x_3 = 3 \\ 3x_1 + 2x_2 - x_3 = 1 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix} , |A| = 24.$$

$$x_1 = 3/8 , x_2 = 3/8 , x_3 = 7/8 .$$

حل السلسلة رقم 04

التمرين 08:

القانون العام للقيم والأشعة الذاتية: $AX = \lambda X$

1/ قانون ايجاد القيم الذاتية:

$$|A-\lambda I| = 0 \vee |\lambda I - A| = 0$$

2/ قانون الأشعة الذاتية:

$$(A-\lambda I)\cdot X = 0$$

مثال: $A = \begin{pmatrix} 5 & -3 \\ -6 & 2 \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -3 \\ -6 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 7\lambda - 8 = 0$$

$$\lambda_1 = 8 \qquad \lambda_2 = -1$$

$(A-\lambda I)\cdot x = 0$ $\qquad \lambda = -1$

$$\begin{pmatrix} 5-(-1) & -3 \\ -6 & 2-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -3 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$6x_1 - 3x_2 = 0 \iff 6x_1 = 3x_2$$

$$x_1 = \frac{1}{2}x_2$$

ليكن $x_2 = t$ ، $t \in \mathbb{R}^*$

$$x_1 = \frac{t}{2}$$

$$\vec{V} = \begin{pmatrix} t/2 \\ t \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} t.$$

$\lambda = 8$

$$\begin{pmatrix} 5-8 & -3 \\ -6 & 2-8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3x_1 - 3x_2 = 0$$

$$-3x_1 = 3x_2 \iff -x_1 = x_2$$

$$x_1 = -t \Leftarrow x_2 = t$$

$$\vec{V} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t.$$

$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ ، $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 3\lambda - 4 = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = 4$$

1/ $\lambda = -1$

$$\begin{pmatrix} 1-(-1) & 3 \\ 2 & 2-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 + 3x_2 = 0$$

$$x_1 = -\frac{3}{2}x_2$$

$$x_1 = -\frac{3}{2}t \qquad x_2 = t$$

$$\vec{V} = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} t.$$

$$D = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ 2 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 + 1\right] = 0$$

$$1-\lambda = 0 \Rightarrow \lambda_1 = 1$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\Delta = -4 = 4i^2$$

$$\sqrt{\Delta} = 2i$$

$$\lambda_2 = \frac{2-2i}{2} = 1-i$$

$$\lambda_3 = \frac{2+2i}{2} = 1+i$$

$\lambda = 1$ /1

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_2 = 0$$

$$x_1 = 0$$

$x_3 = t$ كيفي

$$\vec{V} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t.$$

$\lambda = 4$ /2

$$\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 3x_2 = 0 \qquad x_1 = x_2$$

$$\vec{V} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t.$$

$$B = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix} \quad |A - \lambda I| \Leftrightarrow \begin{vmatrix} -1-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}$$

$$\lambda^2 - \lambda - 6 = 0 \Leftrightarrow (\lambda+2)(\lambda-3) = 0$$

$$\lambda = -2, \quad \lambda = 3$$

$\lambda = -2$ /1

$$\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + 4x_2 = 0 \Rightarrow x_1 = -4x_2$$

$$\vec{V} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} t.$$

$\lambda = 3$ /2

$$\begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4x_1 + 4x_2 = 0 \Rightarrow x_1 = -x_2$$

$$\vec{V} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t.$$

$$G = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda + 3) = 0$

$\lambda_1 = 1 \lor \lambda^2 - 3\lambda + 3 = 0$

$\Delta = -3 = 3i^2, \sqrt{\Delta} = \sqrt{3}\, i$

$\lambda_2 = \dfrac{3 - \sqrt{3}i}{2}, \quad \lambda_3 = \dfrac{3 + \sqrt{3}i}{2}.$

$$F = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ -1 & 2-\lambda & 0 \\ 1 & -1 & 3 \end{vmatrix} = 0$$

$(1-\lambda)\left[\lambda^2 - 5\lambda + 5\right] = 0$

$\lambda_1 = 1 \lor \lambda^2 - 5\lambda + 5 = 0$

$\Delta = 25 \qquad \sqrt{\Delta} = 5$

$\lambda_2 = \dfrac{5 - \sqrt{5}}{2}, \quad \lambda_3 = \dfrac{5 + \sqrt{5}}{2}$

$\lambda = 1 \quad /1$

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 1 - i \quad /2$

$$\begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 2 & -1 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$i x_1 - x_2 = 0 \qquad x_2 = i x_1$

$x_1 + i x_2 = 0 \qquad x_1 = x_1$

$2x_1 - x_2 + i x_3 = 0$

$2x_1 - i x_1 + i x_3 = 0$

$x_3 = (2i + 1) x_1$

$x_1 = t$

$$\vec{V} = \begin{pmatrix} t \\ it \\ (2i+1)t \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ 2i+1 \end{pmatrix} t.$$

$\lambda = 1 + i \quad /3$

$$\begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 2 & -1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-i x_1 - x_2 = 0 \qquad x_2 = -i x_1$

$x_1 - i x_2 = 0$

$2x_1 - x_2 - i x_3 = 0$

$2x_1 + i x_1 - i x_3 = 0$

$(2 + i) x_1 = i x_3$

$x_3 = (1 - 2i) x_1$

$$\vec{V} = \begin{pmatrix} 1 \\ -i \\ 1 - 2i \end{pmatrix} t$$

$\frac{-3-\sqrt{5}}{2}x_1 + x_3 = 0$

$-x_1 - \frac{1+\sqrt{5}}{2}x_2 = 0$

$x_1 - x_2 + \frac{1-\sqrt{5}}{2}x_3 = 0$

$x_3 = \frac{3+\sqrt{5}}{2}x_1$

$x_2 = \frac{1-\sqrt{5}}{2}x_1$

$$\vec{V} = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} \end{pmatrix}$$

التمرين - 2 -

$$(S)\begin{cases} x_1 + x_2 - x_3 = 3 \\ x_2 + x_3 = -1 \\ x_2 + 2x_3 = 4 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 4 \end{array}\right) L_3 = L_3 - L_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right) \begin{array}{l} x_1 = 14 \\ x_2 = -6 \\ x_3 = 5 \end{array}$$

الطريقة الثانية :

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[(1-\lambda)(2-\lambda) - 1\right] = 0$

$1 \vee \lambda^2 - 3\lambda + 1 = 0 \rightarrow \lambda = \frac{3-\sqrt{5}}{2}$

$x_3 = 0$

$-x_1 + x_2 = 0 \qquad x_1 = x_2$

$x_1 - x_2 + 2x_3 = 0$

$$\vec{V_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c.$$

$\lambda = \frac{5-\sqrt{5}}{2}$ /2

$$\begin{pmatrix} 1-\frac{5-\sqrt{5}}{2} & 0 & 1 \\ -1 & 2-\frac{5-\sqrt{5}}{2} & 0 \\ 1 & -1 & 3-\frac{5-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\frac{-3+\sqrt{5}}{2}x_1 + x_3 = 0$

$-x_1 - \frac{1-\sqrt{5}}{2}x_2 = 0$

$x_1 - x_2 + \frac{1+\sqrt{5}}{2}x_3 = 0$

$x_3 = \frac{3-\sqrt{5}}{2}x_1$

$x_2 = \frac{\sqrt{5}+1}{2}x_1$

$$\vec{V} = \begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \\ \frac{3-\sqrt{5}}{2} \end{pmatrix} c$$

$\lambda = \frac{5+\sqrt{5}}{2}$ /3

$$\begin{pmatrix} 1-\frac{5+\sqrt{5}}{2} & 0 & 1 \\ -1 & 2-\frac{5+\sqrt{5}}{2} & 0 \\ 1 & -1 & 3-\frac{5+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$